A response to Bos's *Numbers, Magnitudes, and Ratios*

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Contents

In *Numbers, Magnitudes, and Ratios* [Bos90], Bos briefly outlines the history of the three titular mathematical notions. He also describes the "obstructions" that mathematicians historically faced to viewing these three ideas as being close to "isomorphic". According to Bos, the primary conceptual barrier that was faced by pre-17^{th} century mathematicians was a certain narrowmindedness in mathematical intepretation. Specifically, the way in which mathematicians interpreted numerical operations differed from their interpretations of operations on magnitudes or ratios. The fact that numbers were in the realm of discrete counting while magnitudes involved lengths or weights made the two notions incomparable. While the author agrees with the main contentions of Bos's article, we recognize a strong possibility that a reader of [Bos90] might obtain the false impression that numbers, magnitudes, and ratios are indeed the same notion. Without further clarity, Bos's paper appears to make the claim:

> Up until the 17^{th} and 18^{th} centuries, Mathematicians did not recognize that numbers, magnitudes, and ratios are the same idea, i.e are "isomorphic," and could be treated on equal footing.

The author believes that the above claim is not precise enough, and is slightly incorrect from a mathematical point of view. We assert that Bos most likely meant the following:

> Up until the 17^{th} and 18^{th} centuries, Mathematicians were unaware that numbers, magnitudes, and ratios *could be reconciled* under a common abstract framework. Nevertheless, they

recognized that these three notions had structural differences sufficient to view the ideas as distinct.

The most apparent issue with Bos's main contention is the term "isomorphic." Without going into too much mathematical detail, this statement is false in general. We can say that numbers and rational ratios are *isomorphic as sets* (i.e they effectively have the same size), and that numbers (resp. rational ratios) *inject* into the the set of rational ratios (resp. magnitudes), which means that there is at least one way that we can sit the set of all numbers inside the set of all ratios. The point is that the algebraic properties of numbers, magnitudes, and ratios may be similar, but they are distinct enough to warrant individual study. Indeed this is what happens at the modern research level.

We also recognize the possibility of a modern skeptic not believing Bos's arguments. Bos recognizes that "in most practical purposes, ratios of magnitudes were often represented or approximated by ratios of numbers," and that "the obstructions [to unifying the various operations] were mostly theoretical," yet at the same time this theoretical obstruction may have hindered mathematical progress for thousands of years. A skeptic is most likely to point to the greates minds of the pre–Newtonian era and reason that surely such a theoretical discrepency did not exist in reality. The goal of this present paper is threefold: to provide a summary of Bos's arguments and give additional historical support, to use heuristic examples to counter criticism of Bos's argument, and to offer a modern mathematical retrospective in hopes of illucidating and clarifying Bos's main claim.

1 Summary of the argument

Bos first defines numbers, magnitudes, and ratios. To recap, by a *number* we mean an element of $\mathbb{N} = \{0, 1, 2, 3, \dots\}$[1]. Numbers were the natural realm of counting the physical world, e.g plots of land, bundles of hay, gold coins, planets in the sky, etc. In particular it is clear how and why $\mathbb{N}$ was studied by the Egyptians, Babylonians, Greeks, and Medieval Europeans.

Bos defines a *magnitude* as any *physical* length, area, volume, or weight. Here the word "phyical" roughly means being derived from the physical world. Magnitudes are related to the physics of the universe, and since one could use say – grains of sand – as a medium to define weights, the possible magnitudes seemingly forms a continuum. Mathematically speaking, the only continuum of objects that can be added and well-ordered is the set of real numbers, which we denote by $\mathbb{R}$[2]. Bos also compares magnitudes to real numbers,

[1]For now lets include 0 even though this is a historically contentious number

[2]Historically it is more accurate to use the positive real numbers $\mathbb{R}^+$, but our discussion does not concern negative numbers

though he arrives at the analogy by considering the continuum of possible lengths. One could make the same argument using planar areas or volumes in real 3–dimensional space. Bos argues that *the discrepency of dimensionality*, defined to be the philosophical contention that objects of different kinds (having different units or dimensions) cannot be compared, posed an obstruction to the realization that numbers, magnitudes, and ratios had almost the same arithmetic properties.

Finally, Bos defines a *ratio* as a relation between a pair of magnitudes or a pair of numbers. Moreover, a ratio can only occur between magnitudes of the same kind. With this definition we cannot exactly say that ratios are analogues of what we call the rational numbers (denoted $\mathbb{Q}$) as the Greeks (e.g Euclid and Eudoxus) were aware that "irrational ratios" exist. In retrospect this definition also resembles $\mathbb{R}$. However, the only operation that was recognized between ratios was called "compounding," which was basically the multiplication of real numbers. Ironically, the only operation available to magnitudes was addition of lengths/weights. The combination of these two operations give $\mathbb{R}$ both its multiplicative and additive structures. Though as Bos explains, the compounding of ratios and the addition of lengths were both seen a form of "addition" operation on the relevant space. The fact that addition and multiplication were defined as numerical operations, while only addition was defined for magnitudes and ratios resulted in a general viewpoint that numbers were distinct from magnitudes and ratios. Another observation that separated the philosophical notions of a number versus a magnitude/ratio was that numbers were "discrete" objects, whereas magnitudes and ratios could fill up a continuum.

2 Clarifying the argument

Bos also describes an obstruction to defining the addition of ratios. As we now know, if $a, b, c, d \in \mathbb{R}$, then $\frac{a}{b} + \frac{b}{d} = \frac{ad+bc}{bd}$. However, this definition requires first defining the values ad, bc and bd. This was problematic for early mathematicians when a, b, c, d were magnitudes. For instance, if a and d are lengths, then mathematicians regarded ad as the area of the rectangle with side lengths a and d. To Renaissance mathematicians, the discrepency of dimensionality introduced philosophical issues with this notion of a product. The modern skeptic might argue that a-priori there is no issue here since lengths and areas are clearly both representable by a number, or at least can be approximated by a raio of numbers, and such a fact would surely have been apparent to the great minds of antiquity. To illustrate, we use an example that is modelled on a reasonably likely scenario in ancient Greek/early European society.

so we can proceed without ambiguity

Example 1. Suppose we have line segments $l(a), l(b)$ of prescribed lengths a and b and suppose that $l(a)$ (resp. $l(b)$) can be spanned by laying 2 (resp. 3) "standard Greek bricks", which we define to be a square (or cubic) brick measuring 1 orgyia[3] by 1 orgyia (or $1 \times 1 \times 1$), end to end. The interior of a rectangle with side lengths a and b can be spanned with 6 standard bricks.

This toy scenario makes it seem ridiculous that Greek and early European mathematicians could understand lengths and areas, but could not reconcile the idea that multiplication of lengths results in an area. At the same time, our choice of normalization of lengths via the standard Greek brick is basically Descartes' main insight of coordinate geometry. Bos also points this out, referencing Descartes' own words of "introducing arithmetical terms into geometry." Once we have a notion of an abstract unit square (cube), converting multiplication of magnitudes into multiplication of numbers is natural. For this reason it is necessary that we temporarily forget about coordinate geometry[4]. From here-on-out we work under the assumption that ancient/early–European mathematics was divided into three main, non–overlapping, sects: a calculus of numbers, one of magnitudes, and one of ratios. The calculus of magnitudes was further divided according to *kinds*. A magnitude such as length could be thought of as a number *together with* some additional piece of data. The following examples illustrate two ways that early mathematicians could have thought of magnitudes, the first of which is the scenario suggested by Bos and is historically most accurate. The latter is a fictionalized viewpoint designed to highlight how the ancient's approach to mathematics was convoluted by extra mathematical structure. Unbeknownst to mathematicians, the extra structure was something they apriori imposed on mathematical systems. When we consider the adoption of such meta–mathematical rigidity, it is easy to see how early mathematicians failed to realize a unification of their various calculi.

Example 2.

1. Let a be a magnitude denoting a length. Then a is not a number; but rather, it is a vector (or line segment or interval) in $\mathbb{R}$. Now suppose a and b be two lengths in the plane $\mathbb{R}^2$. In modern notation we could write a and b in vector notation (with some loss of generality) as $a = [\alpha 0]$ and $b = [0 \beta]$. In other words, a is a vector on the x–axis and b is a vector on the y–axis: see figure **??**. Now we are in the position to define the product ab as the *fundamental parallelogram* of the vector $a + b = [\alpha\beta]$, i.e the square whose vertices have coordinates $(0,0), (\alpha, 0), (0, \beta), (\alpha, \beta)$. Using

[3] Ancient Greek unit; one orgyia $\approx$ 1.85m

[4] For further details on Descartes' contribution to the abstraction of maths, see [Och15]

sophisticated notation this procedure can be expressed as a map

$$\begin{aligned} \Lambda(\mathbb{R}^2)^1 \oplus \Lambda(\mathbb{R}^2)^1 &\to \Lambda(\mathbb{R}^2) \\ (a, b) &\mapsto a \wedge b \end{aligned}$$

where $\Lambda(\mathbb{R}^2)$ is called the *exterior algebra of* $\mathbb{R}^2$, Λ^1 is essentially the set of one–dimensional vectors (lengths) and $a \wedge b$ is the "wedge–product" of the vectors a and b, which returns the parallelogram defined by $a + b$. These are well–defined objects and operations in modern geometry, and the wedge–product is essentially a form of "addition" in the exterior algebra. This modern formalism not only reflects the notion of taking products of lengths known to the Greeks and early Europeans, but it also hints towards why pre–17^{th} century mathematicians were inclined to view the product of lengths as an addition operation. Indeed our construction of $a \wedge b$ involved taking the four "fundamental points" defined by $a + b = \left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right]$. With this example in mind it is not entirely fair to cast judgement on the ancients for being too naïve to realize a unification of the operations on numbers and magnitudes. Their veiwpoint that magnitudes carried additional information other than an abstract quantity is simultaneously narrowminded and incredibly foresightful. Narrowminded since the prototypical early mathematician *only* veiwed magnitudes in this fashion, and this agrees with Bos's analysis. But these mathematicians' work is also surprisingly sophisticated once you work out all of the formal details; in particular since much of modern mathematics deals with studying a given object *that is equipped with additional structure*. Early mathematicians simply lacked adequate systems of symbolism and abstraction to realize and understand the high–level mathematical structure that they were studying. Hence Bos's original claim that mathematicians failed to recognize that the operations on numbers, magnitudes, and ratios could be reconciled under a common framework is only part of the picture. The operations amongst numbers, magnitudes, and ratios may obey the same rules, but the global mathematical system of $\mathbb{N}$ is different from $\mathbb{R}$ or $\mathbb{Q}$. An interesting open problem in mathematics is whether there exists an *implicit* definition of $\mathbb{Z}$[5] inside of $\mathbb{Q}$, meaning given a definition of $\mathbb{Q}$ that is independent of $\mathbb{Z}$, can we naturally recover $\mathbb{Z}$ as the subset $\{\cdots, \frac{-2}{1}, \frac{-1}{1}, \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \cdots\}$ of $\mathbb{Q}$ that we usually think of? The fact of the matter is that there are many ways of realizing $\mathbb{Z}$ as a subset of $\mathbb{Q}$, all of which are algebraically isomorphic[6]. Since any implicit definition of $\mathbb{N}$ in $\mathbb{Q}$ would imply one of $\mathbb{Z}$ in $\mathbb{Q}$, to say that $\mathbb{N}$ and $\mathbb{Q}^+$ (i.e numbers and positive rational ratios) are close to being isomorphic

[5] The integers $\{\cdots, -2, -1, 0, 1, 2, 3, \cdots\}$
[6] as rings

only tells part of the story of mathematics, and that their study as fundamentally distinct mathematic objects is a necessary consequence of an abstract axiomatic approach to maths. Of course, these tools were not available to mathematicians until the 17^{th} century[7].

2. In this second example, we imagine that a magnitude a is not simply the length of an interval, the area of a rectangle, or the weight of a physical object; but rather, the value a represents *both* the physical interval (resp. rectangle, bullion) *and* its length (resp. area, weight). In otherwords, a is an element of the set

$$\{\mathcal{R} = \{(r, \alpha) \in R \times \mathbb{R} : r \text{ is a rectangle and } \alpha \text{ is the area of } r\}$$

where R is set of all rectangles and $\mathbb{R}$ is the set of areas. Note that the space $R \times \mathbb{R}$ is 2–dimensional (one dimension of rectangles plus one dimension of numbers). Using Bos's notation, let $r(a, b) = \text{rect}(a, b) \in R$ denote the rectangle with side lengths a and b. The corresponding element in $\mathcal{R}$ is the tuple $(r(a, b), ab)$, where ab is the usual product of real numbers. The observation that $r(a + c, b) = r(a, b) + r(c, b)$, included in the Greek's "geometrical algebra," relfects the distributive property of real numbers. At the level of the space $\mathcal{R}$, this property reads $(r(a+c, b), (a+c)b) = (r(a, b), ab)+(r(c, b), cb)$. The failure of mathematicians to recognize that R is "isomorphic" to $\mathbb{R}$ can be now reinterpreted as a failure to recognize that $\mathcal{R}$ is a one dimensional subspace of $R \times \mathbb{R}$. This is analogous to how the diagonal line $y = x$ in the plane $\mathbb{R}^2$ can be regarded as a single copy of the real line $\mathbb{R}$. This example demonstrates that early mathematicians were making their efforts under the guise of some sophisticated mathematics, which is a problem since they were unaware of this fact. In fact, only with Descartes' coordinate geometry and Leibniz's introduction of symbolism could this problem be solved.

Under the framework of ratios of magnitudes, Bos notes that it is possible to define a closed operation of addition if one interprets $\frac{ad+bc}{bd}$ as "the ratio of line segment lengths that is canonically geometrically deduced from the ratio of areas $ad + bc : bd$". History shows that his idea was not embraced, and Bos observes that up until the 19^{th} century "ratios were not considered primary objects of mathematics, but instead were relations between objects." Perhaps it was this lower–tier status of ratios that obstructed the Greeks from developing a "universal system of mathematics." On a different note, historians of science and mathematicians are in common agreement that mathematicians throughout history have been drawn to *mathematical elegance*, which may be loosely described as a special simplicity of certain mathematical facts. In comparison to the elegance of the Greek's proof that $\sqrt{2}$ is irrational or that the set of prime numbers is

[7] Really one could argue the full force of these tools were not available until the post–Bourbaki era in the 20^{th} century

infinite [c.f Appendix A], the idea of having to temporarily change dimensions in order to define the addition of ratios (e.g $\{1\text{–d lines } a, b, c, d\} \to \{2\text{–d rectangles } ad, bc, bd\} \to \{1\text{–d lines } ad+bc, ad\}$) is non–elegant. We believe that this relative lack of simplicity was an additional barrier for the early adoption of reconciliatory ratio arithmetic. We reserve precisely defining *mathematical elegance* since our belief necessarily implies that mathematicians' understanding of elegance evolves over time. This is possibly a discussion for a future paper.

In summary, Bos's arguments provide a good framework for historians of science to understand exactly how mathematics was impacted by Descartes' coordinate geometry, as well as the algebra and calculus revolutions that followed with figures such as Newton and Leibniz; however, Bos's main argument could easily be misinterpreted or misunderstood. Therefore, it is important that we revisit the discussion with a slightly more modern mathematical veiwpoint so that the fundamental structure of Greek/proto–European mathematics and the reasoning of Greek/proto–European mathematicians can be better understood.

A Appendix

Theorem. *$\sqrt{2}$ is irrational.*

Proof. Suppose towards a contradiction that $\sqrt{2}$ is rational and, when written in lowest terms, is equal to $\frac{p}{q}$ for some integers p and q. Then

$$2 = \frac{p^2}{q^2} \tag{1}$$

so $2q^2 = p^2$. If p were odd, then p^2 would be odd; but $2q^2$ is clearly even, so p is even. Thus there exists $k \in \mathbb{Z}$ so that $p = 2k$. Substituting $p = 2k$ into 1 and rewriting the equation yields $q^2 = 2k^2$; which, by the same argument as before, implies that q is equal to $2j$ for some $j \in \mathbb{Z}$. It follows that $\frac{p}{q} = \frac{2k}{2j}$ is not written in lowest terms, so we arrive at a contradiction. We conclude that $\sqrt{2}$ is not equal to any rational number. □

Theorem. *There are infinitely many prime numbers.*

Proof. Suppose towards a contradiction there are finitely many primes, labelled $p_1, p_2, \ldots, p_N$ and consider the number $X = p_1 p_2 \cdots p_N + 1$. Observe that X divided by any one of the p_i returns a remainder of 1, so X is not divisible by any of the p_i. This means that X itself is a prime number that does not appear in the set $\{p_1, \ldots, p_N\}$, which contradicts the assumption that there are finitely many primes. □

References

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