Rationalism and the Break from Greek Tradition: The Abstraction of Mathematics in the 17th Century

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Abstract

The history of mathematics has tended to be periodic in terms of development; however one era stands out in particular as being the formidable years that culminated into a "mathematical revolution", per se. Although relatively coinciding with, and being influenced by, the Scientific Revolution, this mathematical counterpart is worth investigation in its own right. The scientific overhaul during the 18^{th} century was fueled by the socio-political and philosophical revolutions that were also occurring. In particular there was Descartes' Rationalism, which for nearly a century was at odds with Empiricism, the philosophical ideology underpinning the Scientific Revolution. Although these two philosophical thoughts would become reconciled at the height of the Age of Reason, the dichotomy between Rationalism and Empiricism, naturally beginning with Descartes, would prove sufficient for the formal divergence of mathematical development from the revolution the other sciences were undergoing. This analogous, but fundamentally distinct surge in mathematics had its roots in Descartes' Analytic Geometry, in which Descartes broke away from a fundamental premise of Greek Geometry, liberating mathematics from its strict geometric ties. This "arithmetization" of geometry resulted in the flourishing of abstract mathematics for the next few hundred years alongside science, but ultimately for a different reason. Abstract mathematics resides on the ability to develop a closed system of axioms, upon which mathematics can be built. It very necessarily resided on Rationalism. Thus, not only do we see a connection between Descartes' mathematics and his philosophy, but we also are reaffirmed about the inherent coupling of mathematics with Greek traditions. Additionally, political revolutions in subsequent years were moving towards the foundations of modern Democracy, enhancing the pervasiveness of the

Enlightenment, as indicated by the many contributions to philosophy as well as the creation of the high learning institutions such as the Ecole Polytechnique, which became a boiling pot of mathematical discovery in the 19th century. Moreover, Aristotelian science and philosophy led to Empiricism in the 17th century, the undying philosophy of the Scientific Revolution. Initially at odds with the Rationalist idea that truth can be obtained through reason alone, which instantiated mathematical progress associated with what could be called the "era of abstraction". We see progress such as the notation oriented calculus of Leibniz and the deeply complex ideas of Gauss, each influenced ultimately by Descartes philosophy and mathematics, and each widening the gap between scientific reality and mathematical abstraction. Descartes' Cartesian system gave way to a series of rich discoveries in mathematics; each successive development building off its predecessor, and each ultimately utilizing the tenants of Rationalism to foundationally establish abstraction within the subject (Bell 145), and that is why the mathematical revolution is necessarily different from the overarching Scientific Revolution.

From the era of the great Greek mathematicians up through the Middle Ages mathematical progress in Europe was on the decline. Most mathematicians of these time periods adopted the Greek contention that geometry was the basis of mathematics, i.e. all physical forms can be understood using geometry. This mindset had been inherited from the general Western reverence for ancient Greek society, government, and philosophy following a large period of translating Greek treaties in the 12th century. In fact, it really was this era of translation that set in motion to events for the next five hundred years. The Middle Ages, which experienced the greatest deterioration of mathematical progress, made clear the need for Greek mathematical knowledge to be brought back into the European mathematical world. Overall, the development of mathematics up until Descartes was analogous to the social and political circumstances prevailing during the times. The Dark Ages were the trough of scientific exploration, exemplified by the delicate interplay of conservative government and Christianity (Boyer 194). Europe was isolated following the Christian objects to “pagan science” (Dunham 129). This came to the stagnation of mathematics since no knew knowledge could be elaborated upon. As is seen in history, the Age of Enlightenment immediately following the Middle Ages marked the transition from monarchy to democracy, from agricultural to industrial, from Aristotelism to Rationalism and Empiricism, and as a result of all these various factors, from a mathematical stagnation to an explosion of mathematical realms, concepts, connections, and, above all, abstraction. The social and political changes during this era created an atmosphere conducive to intellectual endeavors, including mathematics. The industrial changes of the 19th century induced a rise in demand for mathematical literacy across all sciences, and allowed mathematicians to more easily share information. Faster methods of travel and the invention of the printing press contributed to the exchange of ideas, leading to the rapid propagation of mathematical knowledge (Boyer 269). Hence, it is historically more productive to treat the mathematical developments following Descartes as separate from those of the Scientific Revolution, specifically the development of abstract mathematics. Even though both mathematics and science flourished immensely during this era due to social and political changes, mathematics was more guided by the tenants of Rationalism and logic, whereas Science couldn’t do without observation.

Part 1: Middle Age Society

The Middle Ages was a time of social decline. The influence of the Royal Family throughout Europe combined with the heightened religious disputes both within Christianity and between the Christians and Muslims during the Holy Wars created an atmosphere characterized by social inequality, religious involvement in government, and not only the neglect of intellectualism, but the full on attack of science and anything else that could be perceived as heresy to the monarch or God as was apparent by the persecution of figures such as Kepler and Galileo for their conjectures about the positions of the heavenly bodies (Bell 279).

Universities in Bologna, Paris, Oxford, and Cambridge were established in the 13th century (Boyer 260). For nearly a hundred years, European mathematics witnessed another small burst in development. The reconstruction of the Greek notion of infinite series, an examination of proportionality, and the first speculations into physical mechanics were explored. Unfortunately, however, the Bubonic Plague would strike most of Europe in the 14^{th} century, and as a result, Europe was sent into another phase of mathematical stagnation. Countries such as France and England would face further socio-political issued in the 15^{th} century with the Hundred Years War and the Wars of Roses, respectively, which undermined the progress of science as a whole, and shifting the center of mathematics from these nations to Italy and Germany during the Renaissance (Bell 95). Although the Crusades had the advantage of breaking down language barriers and thus sending and influx of mathematical knowledge into Europe, the intra-European conflicts resulted in economic, agricultural, political, and moral tolls that the Hundred Years War left in France, and the less severe but similar situation in England during the War of Roses, leaving each respective nation otherwise too preoccupied to delve into mathematical exploration.

Overall, the Middle Ages and Renaissance were considerably more marked as transitional phases in mathematics. This limitation was due to the cultural stagnation and socio-political non-emphasis on scientific progress in the first part of the Middle Ages, and due to the Greek revival experienced in the 12th century, leading to the perpetration of Greek mathematical ideology. Stratification of social classes was not conducive to scientific endeavors either. Even though many prominent mathematicians would show themselves during this time, including Cardan and Napier, the latter being the mathematician to introduce the notion of the logarithm, the full potential of these mathematicians' contributions would not be realized until after the arithmetization of geometry led to a myriad of new possibilities for mathematical exploration; in this case the logarithm's many applications to calculus and number theory, and the questions of solvability of polynomials by square roots ultimate bourgeoning of the immensely powerful Galois Theory. The ideas of Descartes proved to be capable of eventually separating mathematics from the other sciences through the process of abstraction. The Middle Ages in mathematics contained many years of translating Greek and Arabic texts, and so for the majority of those periods, mathematical advancement in Europe rested upon the traditions of the Ancients. The initial steps were to be taken by Descartes with Rationalism and Analytic Geometry, instantiating not only the logic, but also the mathematical inspiration for the next few hundred years.

Part 2: The Greek Dichotomy

Descartes' *Discourse on the Method* not only laid down the foundations of Rationalism, but it simultaneously described a set of tenants that would prevail as the foundations of modern mathematics. In *Discourse on the Method*, Descartes writes "accept as true only what is clear and insusceptible of doubt; divide every problem into as many parts as necessary; consider each part clearly and completely, building by accretion to knowledge of the whole; omit nothing from consideration that might be a source of error" (Descartes Part II). This system, which would become known as the 'Cartesian System," (Rohmann 98) embodied the spirit of mathematical proof (Bell 135). The building blocks of any conjecture require careful examination of every point that offers new information to the problem. This system became more evident in the 19th century as people like Gauss and Cauchy were redefining the notions of mathematical objects and relations, using mathematical abstraction as a tool by which formal proof of claims is achieved (Mehrtens 22). Earlier, though, the independent formulation of calculus by Newton and Leibniz resided upon the exploration into how algebra and geometry are connected, extending upon Descartes mathematics, but utilizing Cartesian logic as a basis (Dunham 163, 190). This highlights the connection between the philosophical ideology of Descartes and his mathematics. More importantly, though, is the way in which Descartes' mathematical break from Greek tradition was linked to his break from Artistotelian philosophy (Rohmann 235). The former paved the way for greater mathematical abstraction, while the latter allowed for mathematics grow through pure reason alone.

Hence Descartes' Analytic Geometry, albeit not entirely mathematically correct, was the missing piece of mathematical machinery that had the power to bring together and expand upon the work of the past few hundred years. This task, not accomplished by Descartes, was motivated by Descartes Rationalism. As Boyer puts it: “From the seventeenth century on, mathematics developed more in terms of inner logic than though economic, social, or technological forces” (335). Number Theory advanced in the subject of Diophantine equations, and Probability Theory rapidly emerged with the new Analytic and Cartesian Geometry. Although these were the immediate repercussions of Descartes math and philosophy, the development of mathematics for the next hundreds of years was fundamentally altered by these contributions, specifically with the idea of “invariance,” which would provide substantial motivation for the development of many ideas, including Riemann’s higher dimensional geometry and ergo Einstein’s’ Theory of Relativity; the former providing an example of the abstraction exemplified by Descartes mathematics, and the latter demonstrating the political influence as well as repercussions procured by his math. Invariance “became of first-rate scientific importance in the 19^{th} and 20^{th} centuries,” (Bell 115) having applications to chemistry, physics, and probability. In this case, however, it was “Descartes epoch-making publication” (101) that gave rise to this importance, and therefore not only do we see mathematics branching away from scientific inspiration, but we in fact have math inspiring science. The history of mathematics still closely followed that of the general Scientific Revolution and Age of Enlightenment, but what distinguished mathematics from here on out was its ability to progress without the need for external scientific motivation. The other Natural Sciences, under their revolution, realized the need for a sufficient system of experimentation to deduce fact, whereas mathematics already had its system of obtaining truth – that is, through Rationalism. Therefore, not only did mathematics exceed the developments of the other science at this point, but it came equipped with a potent logical system of reasoning capable of endless possibilities. The logical system was accompanied by a masterful combination of notation in Descartes’ *Discourse on the Method*. The arithmetization of geometry is apparent in Descartes' math, as the familiar symbols of algebra dominate the pages (Descartes *La Geometrie*). Of most importance is Descartes divergence from tradition Greek interpretations of exponentials. The heart of Greek geometry resides in reducing everything to cubes, squares, and lines, and so something like "x^3" would be interpreted as a cube of unknown length. Descartes, on the other hand, saw this term as a line, just like "x" is a line, except "x^3" is a line of length "x *times* x *times* x". Descartes did not possess an aptitude for mathematical proof, but his philosophical foundations would provide future mathematicians with a means of discerning truth (Rohmann 334). And indeed it was Descartes' Rationalism,

This divergence from the Greek tradition opened the floodgates of mathematical progress, especially later in the 17^{th} Century with the Calculus of Newton and Leibniz. Newton took inspiration from the search for the mechanization of the physical laws, whereas Leibniz, following the Cartesian tenants of Rationalism (Leibniz, Book IV), sought a more general calculus independent of scientific applications. His success was largely due to the fact that Leibniz was "one of the greatest of all notation builders" (Boyer 406). Moreover, Leibniz's take that mathematical truths are "truths of reason and so are *necessary*" (Rohmann 334) showed through in his Descartes-like emphasis on notation, resulting in a larger flexibility of the theory, and "progressing towards a more symbolic theory"(Bell 56). Although Leibniz's contemporaries dismissed this idea of reducing all questions to symbols at the time (Rohmann 418), the continual abstraction and specialization of mathematics led to Cantor's set theory, The Hilbert Program, and Gödel's arithmetization of logical systems in the 19^{th} century (Mehrtens 29, 127). These results owe due to the deep philosophical questions arising out of 17^{th} century philosophy, including Descartes Rationalism. The epistemology insinuated during this era further provoked mathematical questions (Bell 402). Due to the superiority of Leibniz notation in calculus, his formulation saw greater mathematical success in Europe as it arithmeticized infinitesimal analysis. The growth of math would be further fueled by questions arising from Rationalism. In particular we see "Kant's attempt to reconcile Rationalism and Empiricism in the 18th century" (Rohmann 217), which would allow for deep mathematical questions to permeate other natural sciences (Bell 115). We also have the "prolific and popular philosopher Voltaire," (Rohmann 423), who was a large contributor to the philosophical movement of the Enlightenment, being a proponent of "rationalism of Bacon and Descartes" (423). Developments such as these exemplify the divergence of math from the other sciences. Not only do we see mathematics becoming more closely related to a deep philosophical premise of reason, but the socio-political circumstances surrounding these changing philosophical tenants make their way into the development of mathematics, particularly in the political revolutions of Europe.

Part 3: Lasting Affects

Immediate repercussions of Descartes mathematics include the ever so important independent development of the Calculus by Newton and Leibniz. Newton, being the first to officially formulate his theory of fluxions and inverse fluxions, began his investigation into functions by studying Descartes' *La Geometrie* several times over (Dunham 163). The Cartesian system developed by Descartes bridged the gap between geometry and algebra, allowing for a more symbolic approach to the topic, which becomes even more evident in the Calculus of Leibniz, who, influenced not only by the mathematics of Descartes, but also by his rationalist philosophy, set out to "develop a perfect system of formal logic," including a "rational calculus" (Dunham190), thereby reducing the "imprecision of everyday life". Although within mathematics there were heated political debates during this time over who was the true discoverer of the Calculus, what can be said is that Newton failed to publish his results until several years after Leibniz, and moreover, Leibniz's more symbolic calculus revealed deeper questions within the subject (Bell 135). It is also worth noting that Newton's isolation in England during times of great political upheaval, particularly the politicized atmosphere of Trinity College following the Restoration (Dunham 160). This hindered Newton's early career, and was supplemented by the residual fear held by scientists following Galileo's persecution by the Church. This would explain the hesitation, other than the fact that Newton was known to keep to himself (Dunham160). Newton's focus on physical phenomena certainly contributed somewhat to the tardiness of his publications. His intrigue in optics and mechanics developed as a result of his studies of the Cartesian methods, the latter of which being a primary inspiration for his Calculus. Leibniz, on the other hand, was more of a philosopher than a scientist. Not only was he influenced by the mathematics of Descartes, but his formulations are much more resembling of the Rationalist approach provided in *La Geometie,* making them fundamentally more abstract, and hence useful for the next generation of abstract mathematics, which would be dominated by mainland Europe and fueled by the political transformations occurring simultaneously. Here we see the opposite of Newton's situation. The rising philosophical changes occurring in Europe, i.e. the Enlightenment and the Scientific Revolution, was quickly creating an atmosphere of intellectualism, encouraging free thought, and thereby resulting in a surge in the University System, as well as the momentous leap in mathematical abstraction.

The gap between mathematics and science widened following the political revolutions in France and the U.S. In France specifically, the French Revolution was accompanied by the establishment of the Ecole Polytechnique, which became the global center of mathematical development for many years (Mehrtens 57, Boyer 497). Though it was a military academy under Napoleon, the continuous stream of mathematical advancements coming out of the Ecole Polytechnique owe their inspiration to the calculus of Leibniz and Newton, the Number Theory of Fermat (Mehrtens 22), and ergo the Analytic Geometry and Rationalism of Descartes. Many ideas studied in the past lost context in the shadow of the bourgeoning 18th and 19th century mathematics, which surrounded an extensive list of topics ranging from the further formalization of Calculus into Analysis to the study of higher dimensional spaces to the exploration into algebraic structures. All this progress rests on the arithmetization of mathematics. Bell writes "Descartes' invention [Analytic Geometry] suggests useful things to do the in the abstract analysis itself' (137). Math took on a stronger philosophical flavor. Guided by Rationalism, mathematics was free to grow without being restricted by any one mode of interpretation, nor by the demands of science. The mathematical freedom during this era was inherited from the increasing freedom observed in areas such as philosophy and politics. In Germany there was Kant who sought to reconcile Rationalism and Empiricism (Rohmann 218), in France there was the French Revolution, and Mathematicians in the 18^{th} and 19^{th} centuries, including Gauss, took on a different perspective of the concepts of a number. “Numbers were no longer interpreted as objects, but as pure symbols as a means of objectifying mathematical thought – i.e. as a language” (Mehrtens 29). Thus the level of abstraction is seen to have dramatically increased since the time of Galileo and Kepler. This view on numbers is consistent with the Rationalism of Descartes in the sense that Descartes always saw his mathematics as a direct result of his philosophy. Specifically, “I think, therefore I am” (Descartes). This maxim embodies the idea that as long as a concept can be reasoned appropriately and in context, then is has intellectual merit. When this is applied to mathematics during the Enlightenment Era, it says that so long as one can provide a coherent mathematical description, then any notion is plausible (Descartes Part IV). Therefore, the number of possibilities for interpretation grows. Had mathematicians still been confined to the traditional geometric interpretations of the Greeks, then (first of all calculus would most likely not have been discovered for many more years) all mathematics would be the same. There would be no distinction in discipline. Number Theory and Analysis would be fundamentally inseparable. With the introduction of a multitude of mathematical interpretations, sub-categories of math take on their own developmental history – Algebra, Analysis, Topology, Algebraic Geometry, Number Theory, and Logic all began to emerge as distinct disciplines connected through the grand web of mathematics.

The fundamental changes that occurred throughout history that explain the mathematical revolution initiated by Descartes center around two key components: 1. The Greek revival of the Middle Ages and the subsequent break from Greek traditions by Descartes, and 2. The combination of political, social, and philosophical revolutions taking place in the years following Descartes' treaties on Rationalism and Analytic Geometry. It was this combination of factors that led to a mathematical revolution that paralleled its scientific counterpart. As has been seen, there was a distinct set of events in history that motivated the eventual explosion of abstract mathematics, a discipline untied from the endeavors of science and bordering on the lines of philosophy. In fact, it became Descartes' Rationalism that provided the initial framework for mathematics to become a more coherent logical language. Superior mathematical notation was an indirect repercussion of the rationalization and arithmetization of mathematics, allowing for greater flexibility within mathematical exploration, as indicated by the notations of Descartes, Fermat, and Leibniz. Moreover, the political revolutions in Europe during the 18^{th} and 19^{th} centuries generated a more profound atmosphere of nationalism as well as progressivism. The Ecole Polytechnique was opened around the time of the French Revolution and would be responsible for placing France at the center of the mathematical world (Boyer 418). However, systems of higher education also played a strong role in the short burst of mathematical creativity during the upper Middle Ages. The foundation of places like Cambridge and University of Vienna increased the general aura of intellectualism in Europe. What distinguishes mathematics from the other sciences, however, is its unique set of historical ties, i.e. the interplay between Arabic, Greek, and European mathematics throughout history, and the way in which mathematics was realized to be a self-contained subject. New insights could be obtained simply from cutting out the deep-rooted Greek mathematical mindset and putting an emphasis on rigor and coherence. Hence we get the development of the foundations of mathematics, the focus on proofs (Bell 102), and therefore, and endless exploration induced by a philosophical notion and a mathematical breakthrough. Science could not enjoy such a privilege since the foundation of science naturally rests upon observation and experimentation. As a result, the height of the Scientific Revolution would not come until after Empiricism and Rationalism were reconciled and industrialization had already made its impact across the globe (. In this sense mathematics became the purest of all sciences as it was not restricted to anything observable to humanity, but rather, mathematics diverged off and became the study of philosophical reasoning through symbolic abstraction (Bell 115), an embodiment of the philosophical spirit of Rationalism.

Appendix A: Mathematical Overview

Up until the 17th century, the history of mathematic paralleled the history of science. The study of the non-ideal forms within nature boiled down to understanding the ideal forms in mathematics. The Greeks used geometry to deduce the structure of the physical world, the Arabs devised principles of counting and algebraic manipulation to facilitate the transaction of goods, medieval mathematicians built upon Arabic algebra with the study of polynomial equations, and the Renaissance saw an advance of geometry for the use of studying extraterrestrial phenomena. However, it was not until the emergence of Rationalism with Descartes's *Discourse on the Method*, that mathematics did not have to be confined to scientific observation, allowing more freedom for discovery. Descartes's rationalism is apparent in his attempt at analytic geometry, which he described as "the study of geometric questions using algebraic means" (Descartes). Later, Newton and Leibniz would independently discover calculus, expanding on analytic geometry. This unification of disciplines, combined with the masterful notation of Leibniz's calculus ultimately gave rise to the influx of pure abstraction. In particular, it was Descartes's influence on Leibniz's mathematics that allowed people like De'Moive, Taylor, and Euler to find structure within mathematics that did not depend on observation, but rather, with the power of reason, mathematical truth flourished.

Descartes's *Discourse on the Method* outlined the principles by which an individual obtains scientific knowledge. In particular, all knowledge can be deduced from pure, rational thought, specifically with the utilization of mathematical logic (Descartes). In essence rationalism is the opposite of empiricism. It was empiricism that laid the foundations for the scientific method, and the duality of mathematics and science had been undeniably observed in the years leading up to Descartes. A mathematical problem was seen as a means to understand the physical realm of reality. However, with the advancements in mathematics beginning with Descartes, who attempted to uncover the relationship between algebra and geometry via the study of analytic geometry, the world of math and science would slowly start to diverge. Math could be more effectively studied in its own right. Descartes saw little success though, most likely because he still viewed algebraic expressions geometrically (Descartes).

Newton may have "discovered" calculus first, and although principally identical to Leibniz's formulation, it lacked in the power of abstraction. This was partially due to Newton's focus on the empirical, and partially because Leibniz was "one of the greatest of notation builders" (Boyer 406). His use of symbols to represent concepts facilitated the realization of structure inherent to analysis. Furthermore, the extra abstraction induced though symbolism illuminated the overall mathematical utility of his theorems. Newton's calculus was derived substantially from verbal arguments, which left a cloud of obscurity looming over the rich implications of Newton's results (cf. Principia). Leibniz, influenced by Descartes, believed in provocative notation. He coined terms like "function", and produced the symbols for dot multiplication, congruence, and similarity that we use today. His foremost insight was his advocacy for a complete set of symbolic logic upon which all of mathematics could be understood (Leibniz). This stance is saturated in Descartes's rationalism, especially the suggestion that mathematics should be symbolic. At the time this idea was not popularized, but would later be revisited with Cantor's set theory (Rohmann 357). Hence, it was the rationalism of Descartes that would eventually give rise to the abstraction of mathematical logic through symbols, thereby leading to the total abstraction that dominates modern mathematics.

Following Leibniz, it was L'Hopital, a French mathematician and correspondent of Leibniz, who would build off the notation set forth by Leibniz to study the properties of limits, a concept that Newton knew about but did not develop formally. Additionally, infinite series had been studied since James Gregory in the 17th century, but it wasn't until the early 18th century that Brook Taylor would set the foundations for the study of polynomial approximations with the aid of Leibniz notation (Boyer 429). Contributions made by De'Moive during this time also relied on the Leibniz system, and the functional analysis devised by Leibniz enjoyed utility in the further development of algebra as well, beginning primarily with Tschirnhaus in the latter half of the 17th century. Although the techniques involved were not fully developed during this time, it was the emergence of an abstract notational system that re-ignited the study of such questions.

The overall purpose of this paper was to demonstrate how the rationalist approach to knowledge, outline by Descartes, culminated into a system of mathematics independent of empirical observation, which was formally initiated with Leibniz's notation in calculus, and propagated through the realization of structure within said notation.

Mathematics is an old discipline embedded within a rich history of expansions and contractions of ideas. The Greeks believed in the power of geometry. The Arabs were responsible for counting principles and algebra. The mathematicians of medieval Europe reconciled geometry with the idea of infinity. The Renaissance was an era rich with mathematical mile-stones, ranging from the adoption of Hindu numerals and German notation for operations to the first glimpses at the pure abstract study of the roots of polynomial equations. The inherent focus on the physical world, individual observation, and the applicability of mathematics to explaining physical phenomena restricted math. That is until Rationalism spread across Europe. Newton's calculus stemmed from observation, whereas Leibniz viewed mathematics as a self-contained system that can be understood through good notation and rational procedures. Symbolic mathematics reflects the nature of abstraction. The scribbles on the page don’t have inherent observability, but they are given power through reason.

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