# The Cave: History's Most Correct Bad Mathematics

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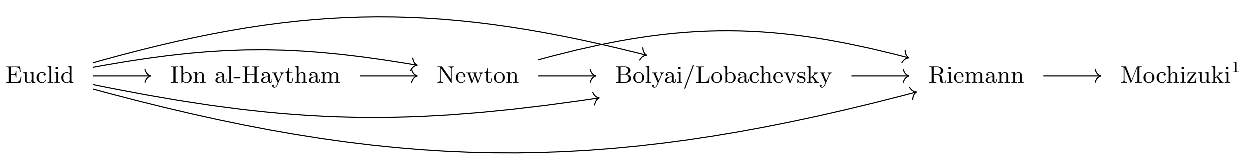
**Abstract**

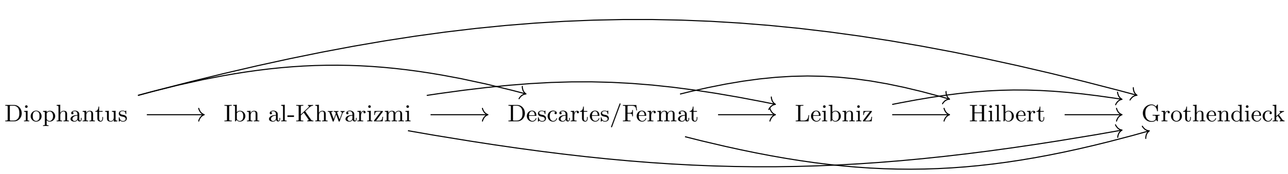
It is generally accepted that the dominant mode of mathematical reasoning up until the $17^{th}$ and $18^{th}$ centuries was that of "mixed" mathematics. We provide evidence that the overall history of mathematics is too non–linear to make such definite claims, and that mixed and pure mathematics have coexisted in a perfect circumstantial balance since antiquity. We outline the primary chronology of non–Euclidean geometry and algebraic geometry to support our standpoint. In particular, while it is true that Greek and Arabic mathematicians were focused on using math to solve practical questions, the requisite progress in their mathematics to solve such problems were always advances of the theoretical sort. We apply our viewpoint to critique and clarify Bos's article *Numbers, magnitudes, and ratios.*

# 1 Introduction

In ca. 380 BCE, Plato's *Allegory of the Cave* imagined a world whose inhabitants are confined to a cave. The prisoners, who are chained to a wall facing away from the entrance, are unable to experience or see the true nature of reality. All that they know is derived from a fire behind them, which projects shadows of the real world onto the wall in front of them. The allegory generally encapsulates the notion of *the ignorance of man*, whereby a human being can never experience the raw, unadulterated state of the Universe. This is especially true when it comes to our ability to reason and understand the mathematical machinations of our world. Historically, scientists as a whole have been locked in a constant battle against their own senses. Indeed the Kantian philosophy of limitations of intuition, discuseed in Kant's *Critique of Pure Reason* echo this doctrine. As history has played out, science almost always prevails over such limitations via steady progress. However, most scientific schools of thought (e.g medicine, biology, chemistry, and physics) follow a progression that is more-or-less linear, with each moment of genius following its immediate successor. All too often the history of mathematical revolution is assumed to follow a similar quasi–linear path. To the contrary, we argue that mathematical history is *highly non–linear*, and history frequently shows that mathematicians need to constantly revisit and revise first (i.e old) principles in order to make substantive strides towards the next wave of (new) knowledge. Two main veins of progress that illustrate this contention are the history of non–Euclidean geometry and the history of the algebraization of geometry. The chronology of major developments in these subjects can be roughly visualized in the following schematics:




(1)




(2)

Compare these diagrams with the contemporarily accepted linear development of the atomic model:

$$\text{Democritus} \longrightarrow \text{Dalton} \longrightarrow \text{Thompson} \longrightarrow \text{Rutherford} \longrightarrow \text{Bohr} \longrightarrow \text{Schrödinger}$$

An arrow $A \to B$ is used to indicate that mathematician $B$ needed to revisit and revise the principles first developed by $A$ in order to overcome certain theoretical obstructions. Diagrams 1 and 1 stand in contrast to the linear way in which mathematical history is typically presented. While not the main subject of the present work, the author believes that the non–linearity of mathematical history is primarily caused by the notion of "proof", an idea that is unique to mathematics. We plan on studying this claim in a future paper. Presently we focus on the *consequences* of viewing mathematical history as a linear timeline. Specifically, it is commonly accepted that "mixed" mathematics [Bro91, EKSS13, Bro15] was the predominant mode of mathematical science up until the $17^{th}$ and $18^{th}$ century; and only with Newton's method of fluxions and Leibniz's symbolic infinitesimal calculus does one see a divergence of mixed and theoretical mathematics. Our argument will show that this viewpoint not only downplays significant historical developments in pure mathematics, such as Al-Khwarizmi's abstract algebraization of geometry, but it also does not take into consideration the reoccurring dichotomy between the *subject's* (i.e the mathematician) ability to do and understand math, and the mathematical *objects* under study themselves. For future reference, we summarize some of these quarreling states in table 1.

Table 1: Summary of Human (Mathematical) Limitations

| Subject | Object |
|---|---|
| Mathematician | Mathematics |
| Language | Ideas |
| (Perception of) Unreason | (Existence of) Reason |
| Intuition | Raw state of the Universe |

The last two rows of the table above deal primarily in philosophical issues, which we will not consider in the present work. On the other hand, the dichotomy presented in the first two rows can, and should, be interpreted as the driving force behind mathematical progress throughout the ages, and not just during the $17^{th}$ and $18^{th}$ centuries. Moreover, the perennial manifestation of technical and philosophical issues obstructing mathematical progress can be linked, directly or indirectly, to a fundamental chasm that exists between a mathematician's abilities, limitations, and available toolkit, and the mathematical ideas that he or she is trying to uncover. Again, we note that this phenomenon is rarely present in other sciences since the primary non–mathematical scientific tool is experimentation and observation. As we will show, the divide between subject and object within mathematics not only explains the parallel development of the mixed and theoretical versions, but it simultaneously perpetuates mathematicians' need to revisit old principles in order to make new progress. This what we mean by the titular terminology of "correct bad mathematics". Arguably the most famous recent instance of revisiting old principles is Alexander Grothendiecks invention of *algebraic geometry* [Gro67]. In his 8–part series, totally over 1500 pages of pure (hard) abstract mathematics, Grothendieck effectively reimagines the study of Geometry and collects many mathematical analogies that had been developed over the past two-thousand years under a single common theory.

To illustrate why our reinterpretation of mathematical history is significant, we will revisit and clarify the issue of *Numbers, Magnitudes, and Ratios* described by Bos [Bos90]. In particular, a non–linear viewpoint of math's history justifies our critique of Bos's stance on the certain inability for mathematicians to recognize the "isomorphisms" between various systems of arithmetic. We appeal to a retrospective analysis, drawing upon developments occurring before and after the $18^{th}$ century that refute the idea that, say, numbers and magnitudes are the same object. At the same time, we revise the traditionally accepted strict distinction between the history of "mixed" and theoretical mathematics. In fact, we argue that the two modes of mathematics have always coexisted, with the practical applicability of Greek and Medieval-European mathematics merely overshadowing the requisite theoretical underpinnings. At the same time, we show that whenever the theoretical side is obstructed, the utilization of less rigorous practical mathematics contributes to the reconciliation of theoretical obstructions.

The structure of the paper is as follows: in sections 2 and 3 we recall the history of Greek and Arabic contributions to mathematics. We contrast the traditional view that focuses on the preponderance of mixed mathematical methods with our emphasis on the Greek and Arabic contributions to theoretical math. Section 4, the main component of the present work, studies the mathematical revolution occurring in $16-17^{th}$ century Europe in the context of the Greek and Arabic contributions. We also highlight, using post-$17^{th}$ century developments, why the Newtonian and Leibnizian revolutions are but one piece of the grander historical attempts to reconcile the columns in table 1. We conclude with our response to Bos's *Numbers, magnitudes, and ratios.*

## 2 Greek Beginnings

Early mathematicians in Greece[2] were mostly concerned with what has typically been referred to as practical mathematics [GK07] in the sense that the tools used and subject matter studied were focused towards

[2] and Egypt

practical ends, e.g surveying land, distributing goods and services, and understanding the heavens. This mode of mathematical reasoning, which would become known as "Aristotelian" or "mixed" mathematics, heavily blurred the line between the subject and object in table 1. While several scholars have argued that mixed math was the main form of reasoning employed by mathematicians up until the late $16^{th}$/early $17^{th}$ centuries, this viewpoint neglects several historically significant developments in pure math, regardless of whether or not such advances were consciously made. Even though the formally understood distinction between pure mathematics and "mixed mathematics" would not be written down until Bacon's *Of the Procience and Advancementof Learnings* (1605) and *De Dig-nitae et Augmentis Scientiarum* [Bro15] (1623), Thales (640 BCE – 546 BCE), Eudoxus (408 BCE – 355 BCE), and Theaetetus of Athens (417 BCE - 369 BCE) were amongst the first to write down the modern equivalent of Theorems. At around the same time, Aristotle (ca. 350 BCE) advocated for definitions based upon prior knowledge. This seemingly innocent request anticipated Euclid's (ca. 300 BCE) axiomatization of geometry via his five postulates [Boy91]. The fifth postulate – the parallel line postulate – contributes to the cyclic nature of mathematical development. Troubled by the seeming complexity of the fifth postulate, Ibn al-Haytham (ca. 1000 AD) attempted to prove it from the other four by means of contradiction, and in the process formulated several Theorems on the structure of quadrilaterals [Ati88]. Al-Haytham's results are regarded as the amongst the first Theorems concerning hyperbolic and elliptic geometries. Newton's methods of fluxions [Gui04, Gui90] involved the use of Greek inspired geometric techniques to further develop the theory of elliptic transcendental curves. While Newton's motivation was primarily to understand the mechanics of celestial bodies, these developments should be viewed as theoretical advancements towards non–Euclidean geometry. Along similar lines, Bolyai and Lobachevsky (ca. 1830 AD) independently appealed directly to Euclid's work and developed the first formal system of non–Euclidean geometry by completely removing the fifth postulate and starting from scratch. Their work, which required looking backwards in time by nearly 2000 years, is a very clear instance of the non–linear nature of mathematical development (see figure 1 for reference). Riemann followed a similar path, revisiting Euclid's *Elements* and Newton's *Principia* in 1858 to develop what we now know as Riemannian geometry, which can be loosely understood as the method of fluxions applied to non–Euclidean curved surfaces.

The shortcoming of Euclid was that his *Elements* introduced definitions first, axioms second, and Theorems last [GK07]. In modern times we understand that the order of the first two items should be reversed. Hence we see one of the first major instances of "good" incorrect pure mathematics in history. Additionally, if we accept the claim that mathematical "proof" is a key feature of pure mathematical reasoning, then the

Greeks were the first society to employ this mode of mathematics. However, the Greeks' notion of mathematical "proof" was relatively weak and only existed in the form of geometric demonstrations [EKSS13]. The failure of geometric demonstrations to capture important mathematical ideas in the eras that followed greatly necessitated the advent of more sophisticated methods. Hence the limitations of Greek mathematics not only serve has the prototypical example of "correct bad (pure) mathematics," ultimately necessitating the geometric algebra of Al-Khwarizmi and Descartes, but their confounding of human sensibilities with inherent mathematical truth prevented them from recognizing the philosophical distinction between columns in table 1. The Greeks went so far as to completely separate the notions of "constructibility" and "proof". Bos notes in [Bos84, II.2] that

> The solution of a problem consited of the construction itself, together with the proof of the validity of the construction.

Today, we understand the above Greek method as a "constructive proof". The existence of non-constructive proofs, e.g ontological arguments, created further disparities between Greek methods and those utilized in the $17^{th}$ and $18^{th}$ centuries. As Boyer notes, a predominately geometric approach would likewise be adopted in Diophantus' *Arithmetica* (ca. 250 AD). Despite this fact, Diophantus is often called "the father of algebra" – a title that Al-Khwarizmi (ca. 820 AD) arguably deserves more: see section **??**. In general, the Greeks' focus on geometric demonstrations influenced their obsessed with classifying geometric quantities [Bos90]. The two–part nature of the Greek's geometric mathematics described in the quote above is typically regarded as a fundamental obstruction to theoretical progress, or is even used to point to the total absence of theoretical considerations [Och15]. However, Greek mathematicians, like their Renaissance European counterparts, we expert philosophers. This observation motivates the claim that the Greeks naturally felt compelled to classify numerical quantities based on their constructivist (or lack thereof) properties. For example, the Greek's realization that certain numbers were irrational would motivate Dedekinds formal construction of the real numbers [Mul16]. Similarly, the Greek's understanding that certain irrational numbers were "constructible" via geometric methods, while others were not, would coalesce into Galois' invention of the Galois theory of equations [Pie98]. Therefore it is naïve to say that theoretical mathematics was not particularly developed until the age of Newton and Leibniz. It is necessary that historians of science recognize that the Greek's considerations of irrationality and constructability are inherently theoretical in nature. It just so happens that the Greeks, being *subjects* (the first column of table 1), had a lack of tools and historical precedent to understand that what they had uncovered was deeply theoretical. The best philosophical tools that they had lead to their ultimate classification of various arithmetic quantities, which is outlined in the main paper of contention [Bos90]. This ultimately hindered Greek mathematical progress since there is an obvious limitation of mathematics that one can do with *sensible* geometric constructions. Queue the Islamic Golden Age.

## 3 Arabic Wisdom

Grothendieck was by far not the first mathematician to algebraize geometry. Al-Khwarizmi – Islamic Scholar, Arabic mathematician, and one of the library directors at the Baghdad House of Wisdom – was likely the first mathematician to seriously use algebraic symbols and methods to deduce geometric principles, the most famous being the solution to the quadratic equation. Indeed, R. Rashed and Angela Armstrong [Ras94] note that

> Al-Khwarizmi's text can be seen to be distinct not only from the Babylonian tablets, but also from Diophantus' Arithmetica. It no longer concerns a series of problems to be resolved, but an exposition which starts with primitive terms in which the combinations must give all possible prototypes for equations, which henceforward explicitly constitute the true object of study. On the other hand, the idea of an equation for its own sake appears from the beginning and, one could say, in a generic manner, insofar as it does not simply emerge in the course of solving a problem, but is specifically called on to define an infinite class of problems

Although Al-Khwarizmi was an astronomer by trade (hence he would traditionally fall under the guise of a mixed–mathematician), his development of algebraic techniques and consideration of generic equations, regardless of the problem they may be used to solve, is in stark contrast to the geometers of the $18^{th}$ century, who "advocated a view of how to apply mathematics to solve problems in areas that included mechanics, astronomy, and the moral sciences" [Bro91]. This supports our claim that Al-Khwarizmi's algebra should be more appropriately characterized as an advancement in pure mathematics. We emphasize that the primary or immediate utility of new mathematics should not be confused with the underlying advancement. Similar to the non–linearity of the history of non–Euclidean Geometry, Al-Khwarizmi's geometric algebra would enjoy multiple resurges for the next thousand years. Descartes' analytic geometry, what we now call *Cartesian geometry*, was undoubtedly influenced by the works of Al-Khwarizmi; specifically those that were translated by Leonardo de Pisa [Ras14]. Likewise, Leibniz's symbolic infinitesimal calculus (and his philosophy) were also inspired by Cartesian geometry (and philosophy, ergo Kantian philosophy as well) [Och17,Och15]. David Hilber's education at the University of Göttingen in $19^{th}$ century Germany brought him naturally under the influence of philosophers such as Kant, as well as mathematicians such as Leibniz [CJ90]. We also remark that there should be an arrow Euclid → Hilbert since, following the set theory advances of Cantor[3], the Hilbert axioms were necessitated as a new *formal* set of geometric Axioms to replace Euclid's five axioms. Such a formalization of geometry paved the way for his famous set of problems introduced at the turn of the $20^{th}$ century; and moreover, these axioms would inspire generations of mathematical progress, including Grothendieck's algebraic geometry revolutions [Acz06].

Arabic mathematics would make its way to Europe through Italy in the $13^{th}$ century, roughly coinciding with the Mongol's siege of Baghdad and the destruction of the Baghdad house of Wisdom [Boy91]. Subsequently, Italy would experience an explosion of scientific activity [Goo91]. As is illustrated by the following passage, supplied by an unnamed source and relayed to the author through Professor Guicciardini, multiple pre–Renaissance European mathematicians and astronomers benefitted immensely from the generic, abstract, and symbolic power of Al-Khwarizmi's algebra.

> Numbers in medieval Arabic arithmetic and algebra encompass any positive quantity that can arise in calculation, including fractions and irrational roots. The utility of these continuous numbers in mensuration guaranteed their persistence throughout the medieval period and across to Europe, despite their incompatibility with the concept of number in Aristotle, Euclid, and Nicomachus. It is because of the influence of this practical tradition that numbers are assigned to lines and areas in many sixteenth-century scholia of Greek books. For Euclids Elements these include translations edited by Jacques Peletier (1557), Pierre Forcadel (in French, 1560), Xylander (in German, 1562), Franois de Foix Candale (1566), Federico Commandino (1572), and Christoph Clavius (1574). Irrational numbers naturally arise in these calculations... Franciscus Barociuss translation of Procluss commentary on Euclids first book (1560) and Commandinos edition of the works of Archimedes (1558) also contain several arithmetizations. Joannes Baptista Memuss calculations in his translation of Apolloniuss Conics (1537) extend to include algebra on many pages, and Johann Scheubels translation of Euclids Elements (1550) begins with a 76-page introduction to algebra, and numerical and algebraic calculations are found scattered among Euclids propositions. Like in Pappus, the numbers in these scholia are used to illustrate the relative sizes of the magnitudes, and were not intended to reflect any absolute system of measurement. This relational nature of the numbers was always the case in trigonometry, too.

To summarize the above passage and this subsection, the practical assignment of numbers to lengths and

[3] Again, inspired by the Leibnizian symbolic and philosophical revolution

areas (i.e *magnitudes*) in medieval Europe was admitted by Al-Khwarizmi's abstraction of symbolic equations. Apriori however, the incompatibility of "continuous" magnitudes with the arithmetic quantities considered by the Greeks fueled the $16^{th}$ century European viewpoint that numbers, magnitudes, and ratios should still be considered as distinct objects.

# 4 Europe: Necessitating Reimagination

Recall that many medieval European mathematicians we master philosophers (e.g Adelard of Bath, Nicole Oresme, Descartes). Note that the latter is the second major figure in the algebraization of geometry. The same phenomena carries over into the $17^{th}$ and $18^{th}$ centuries; most notably with the likes of Leibniz. We also recall Professor Guicciardini relaying the following anecdote concerning Leibniz's correspondence with his student, L'Hôptial:

> Do not worry about the proof for I am the philosopher, ergo I know what is correct.

It is highly likely that these philosopher/mathematician hybrids understood that there *were* fundamental structural differences between numbers, magnitudes, and ratios. However, the distinction between mixed and pure mathematics arising during the $16^{th}$ and $17^{th}$ century was skewed by Bacon's "Tree of Knowledge," which placed geometry and arithmetic in the category of pure math and astronomy, music, perspective, and engineering in the realm of mixed math [Bro91]. During the $18^{th}$ and $19^{th}$ centuries, the notion of mixed mathematics would evolve into that of "applied mathematics" [EKSS13]. The terminology "applied" was, and still is, used to indicate the use of pure methods to solve practical problems. It follows that mixed mathematics from antiquity to the $18^{th}$ century was always a form of applied mathematics. Even so, the consensus amongst contemporary historians of science is that mixed mathematics was the predominant mode of all math up until the split induced by the works of Newton and Leibniz. Overall, not enough consideration is put into the fact that any new development or theory within geometry or arithmetic, regardless of whether it was designed to solve pure or applied problems, is itself advancement in pure mathematics. This is amongst the fundamental reasons why one should not accept linear timeline of mathematics.

This bias towards the effects of $17 - 18^{th}c.$ mathematics on the history of the pure subject is further compounded by the massive influx of $16^{th}$ century physics arising out of the works of Kepler, Copernicus, and Galileo. But, as illustrated in the quotations from subsection 3 and [Gui90, 3.2.2], the algebraic techniques admitting the advances of these figures is directly traceable back not only to Descartes' algebraization of geometry, but also Al-Khwarizmi's. When one considers Newton and Leibniz's contributions to mathematics, e.g calculus, it is known that their techniques were largely geometric [Gui90], despite Leibniz's invention symbolic differential algebra. Indeed the algebraization of differential calculus would not become popular until L'Hôpital and Legendre fixed the sub–rigorous indeterminacies arising in Leibniz's formulation of infinitesimal calculus and Euler formally introduced the notion of a function [Och17]. However, again it would be naïve to think that functional calculus became the dominant mode of mathematics up until contemporary times. As noted in the introduction, Grothendieck's algebraic geometry recast the discipline in an entirely new light. Between Euler and Grothendieck, most mathematicians indeed used the functional approach to calculus. For example, the circle (resp. the hyperbola) was thought of as the points $(x, y)$ satisfying the formula $x^2 + y^2 = 1$ (resp. $x^2 - y^2 = 1$). In other words, half of the circle (resp. hyperbola) could be characterized as a function $y = \sqrt{1 - x^2}$ (resp. $y = \sqrt{x^2 - 1}$). However, prior to Euler, one would typically regard the circle or hyperbola as an "irreducible" mathematical object (instead of being described by the two parameters $x$ and $y$). Similarly, Grothendieck's revolution in the 1950s brought mathematics back into this mindset, understanding the circle as the singular object $\mathrm{Spec}\mathbb{R}[x, y]/(x^2 + y^2 = 1)$. The two variables are present in the "polynomial ring" $\mathbb{R}[x, y]$ (2–dimensional "thing"), but when we "divide out" by the equation $x^2 + y^2 = 1$ (1–dimensional "thing"), we effectively reduce the argument $\mathbb{R}[x, y]/(x^2 + y^2 = 1)$ to a 1–dimensional object. The fact that these resurgences in thought have continued to occur throughout the history of mathematics, even in modern times, introduces additional support towards our claims of the non–linearity of mathematical history, the inability to characterize one era of mathematics as being dominated by "mixed" or "pure" methods, and the existence of a math–specific human obstruction induced by the chasm illustrated in table 1.

## 5 Conclusion: lasting effects

The evidence presented in sections 2, 3, and 4 suggest we cannot simply apply the hisotrical analysis of natural sciences, such as physics, to developments in mathematics from Antiquity to the $18^{th}$ century. While there is little doubt that Newtonian and Leibnizian calculus were insurmountably important towards our modern understanding of mathematics, the sum of the subject's history prior to the $18^{th}$ century is equally significant. Moreover, it is unfair and highly Eurocentric to suggest that the Newtonian and Leibnizian mathematical revolutions were the primary factor contributing to the major split between mixed and pure modes of mathematical reasoning. For one, the introduction of Theoremata, axiomata, and mathematica definition by ancient Greek mathematicians, together with the abstract algebra pioneered by Al-Khwarizmi are clearly characterizations of advances in pure mathematical thought. Further, there were multiple significant developments occuring after Newton and Leibniz's time that ultimately required a revision of mathematicians' understanding and interpretation of Newtonian and Leibnizian mathematics, just like Newton and Leibniz recognized the need to revise their own understanding of the collective works of Wallace, Descartes, Fermat, and Leondardo of Pisa. So it seems that the story of mathematics is akin to the infinite iteration of Plato's *Allegory of the Cave*: we mathematicians are destined to perpetually leave our cave of darkness, only to find ourselves in a slightly larger, more well-lit cave.

## 6 A response to Bos

Assuming the highly non–linear timeline of mathematics argued above, and assuming that the story of mixed and pure methods in mathematics are deeply interconnected, we move on to our critique and clarification of Bos's paper *Number, magnitudes, and ratios* [Bos90]. It is important that the distinction between mixed and pure mathematics is blurred in the discussion that follows. Otherwise, it is easily argued that Bos's argument only applies to those mathematicians who were concerned with the pure, philosophical underpinnings of the subject.

In his paper, Bos briefly outlines the history of the three titular numerical notions. He also describes the "obstructions" that mathematicians historically faced to viewing these three ideas as being close to "isomorphic". According to Bos, the primary conceptual barrier that was faced by pre-$17^{th}$ century mathematicians was a certain narrow-mindedness in mathematical interpretation. Specifically, the way in which mathematicians interpreted numerical operations differed from their interpretations of operations occurring on magnitudes or ratios. The fact that numbers were in the realm of discrete counting while magnitudes involved lengths or weights made the two notions incomparable. While the author agrees with the main contentions of Bos's article, we recognize a strong possibility that a reader of [Bos90] might obtain the false impression that numbers, magnitudes, and ratios are indeed the same notion. Without further clarity, Bos's paper appears to make the claim:

> Up until the $17^{th}$ and $18^{th}$ centuries, Mathematicians did not recognize that numbers, magnitudes, and ratios are the same idea, i.e are "isomorphic," and could be treated on equal footing.

The author believes that the above claim is not precise enough, and is slightly incorrect from a mathematical point of view. We assert that Bos most likely meant the following:

> Up until the $17^{th}$ and $18^{th}$ centuries, Mathematicians were unaware that numbers, magnitudes, and ratios *could be reconciled* under a common abstract framework. Nevertheless, they recognized that these three notions had structural differences sufficient to view the ideas as distinct.

The most apparent issue with Bos's main contention is the term "isomorphic." Without going into too much mathematical detail, this statement is false in general. We can say that numbers and rational ratios are *isomorphic as sets* (i.e they effectively have the same size), and that numbers (resp. rational ratios) *inject* into the the set of rational ratios (resp. magnitudes), which means that there is at least one way that we can sit the set of all numbers inside the set of all ratios. The point is that the algebraic properties of numbers, magnitudes, and ratios may be similar, but they are distinct enough to warrant individual study. Indeed this is what happens at the modern research level.

We also recognize the possibility of a modern skeptic not believing Bos's arguments. Bos recognizes that "in most practical purposes, ratios of magnitudes were often represented or approximated by ratios of numbers," and that "the obstructions [to unifying the various operations] were mostly theoretical," yet at the same time this theoretical obstruction may have hindered mathematical progress for thousands of years. A skeptic is most likely to point to the greatest minds of the pre–Newtonian era and reason that surely such a theoretical discrepancy did not exist in reality. The goal of this present paper is threefold: to provide a summary of Bos's arguments and give additional historical support, to use heuristic examples to counter criticism of Bos's argument, and to offer a modern mathematical retrospective in hopes of elucidating and clarifying Bos's main claim.

## 6.1 Summary of the argument

Bos first defines numbers, magnitudes, and ratios. To recap, by a *number* we mean an element of $\mathbb{N} = \{0, 1, 2, 3, \dots\}$[4]. Numbers were the natural realm of counting the physical world, e.g plots of land, bundles of hay, gold coins, planets in the sky, etc. In particular it is clear how and why $\mathbb{N}$ was studied by the Egyptians, Babylonians, Greeks, and Medieval Europeans.

Bos defines a *magnitude* as any *physical* length, area, volume, or weight. Here the word "phyical" roughly means being derived from the physical world. Magnitudes are related to the physics of the universe, and since one could use say – grains of sand – as a medium to define weights, the possible magnitudes seemingly forms a continuum. Mathematically speaking, the only continuum of objects that can be added and well-

[4]For now lets include 0 even though this is a historically contentious number

ordered is the set of real numbers, which we denote by $\mathbb{R}$[5]. Bos also compares magnitudes to real numbers, though he arrives at the analogy by considering the continuum of possible lengths. One could make the same argument using planar areas or volumes in real 3–dimensional space. Bos argues that *the discrepancy of dimensionality*, defined to be the philosophical contention that objects of different kinds (having different units or dimensions) cannot be compared, posed an obstruction to the realization that numbers, magnitudes, and ratios had almost the same arithmetic properties.

Finally, Bos defines a *ratio* as a relation between a pair of magnitudes or a pair of numbers. Moreover, a ratio can only occur between magnitudes of the same kind. With this definition we cannot exactly say that ratios are analogues of what we call the rational numbers (denoted $\mathbb{Q}$) as the Greeks (e.g Euclid and Eudoxus) were aware that "irrational ratios" exist. In retrospect this definition also resembles $\mathbb{R}$. However, the only operation that was recognized between ratios was called "compounding," which was basically the multiplication of real numbers. Ironically, the only operation available to magnitudes was addition of lengths/weights. The combination of these two operations give $\mathbb{R}$ both its multiplicative and additive structures. Though as Bos explains, the compounding of ratios and the addition of lengths were both seen a form of "addition" operation on the relevant space. The fact that addition and multiplication were defined as numerical operations, while only addition was defined for magnitudes and ratios resulted in a general viewpoint that numbers were distinct from magnitudes and ratios. Another observation that separated the philosophical notions of a number versus a magnitude/ratio was that numbers were "discrete" objects, whereas magnitudes and ratios could fill up a continuum.

### 6.2 Clarifying the argument

Bos also describes an obstruction to defining the addition of ratios. As we now know, if $a, b, c, d \in \mathbb{R}$, then $\frac{a}{b} + \frac{b}{d} = \frac{ad+bc}{bd}$. However, this definition requires first defining the values $ad, bc$ and $bd$. This was problematic for early mathematicians when $a, b, c, d$ were magnitudes. For instance, if $a$ and $d$ are lengths, then mathematicians regarded $ad$ as the area of the rectangle with side lengths $a$ and $d$. To Renaissance mathematicians, the discrepancy of dimensionality introduced philosophical issues with this notion of a product. The modern skeptic might argue that a-priori there is no issue here since lengths and areas are clearly both representable by a number, or at least can be approximated by a ratio of numbers, and such a fact would surely have been apparent to the great minds of antiquity. To illustrate, we use an example that is modeled on a reasonably likely scenario in ancient Greek/early European society.

[5]Historically it is more accurate to use the positive real numbers $\mathbb{R}^+$, but our discussion does not concern negative numbers so we can proceed without ambiguity

*Example* 1. Suppose we have line segments $l(a), l(b)$ of prescribed lengths $a$ and $b$ and suppose that $l(a)$ (resp. $l(b)$) can be spanned by laying 2 (resp. 3) "standard Greek bricks", which we define to be a square (or cubic) brick measuring 1 orgyia[6] by 1 orgyia (or $1 \times 1 \times 1$), end to end. The interior of a rectangle with side lengths $a$ and $b$ can be spanned with 6 standard bricks.

This toy scenario makes it seem ridiculous that Greek and early European mathematicians could understand lengths and areas, but could not reconcile the idea that multiplication of lengths results in an area. At the same time, our choice of normalization of lengths via the standard Greek brick is basically Descartes' main insight of coordinate geometry. Bos also points this out, referencing Descartes' own words of "introducing arithmetical terms into geometry." Once we have a notion of an abstract unit square (cube), converting multiplication of magnitudes into multiplication of numbers is natural. For this reason it is necessary that we temporarily forget about coordinate geometry[7]. From here on out we work under the assumption that ancient/early–European mathematics was divided into three main, non–overlapping, sects: a calculus of numbers, one of magnitudes, and one of ratios. The calculus of magnitudes was further divided according to *kinds*. A magnitude such as length could be thought of as a number *together with* some additional piece of data. The following examples illustrate two ways that early mathematicians could have thought of magnitudes, the first of which is the scenario suggested by Bos and is historically most accurate. The latter is a fictionalized viewpoint designed to highlight how the ancient's approach to mathematics was convoluted by extra mathematical structure. Unbeknownst to mathematicians, the extra structure was something they apriori imposed on mathematical systems. When we consider the adoption of such meta–mathematical rigidity, it is easy to see how early mathematicians failed to realize a unification of their various calculi.

*Example* 2.

1. Let $a$ be a magnitude denoting a length. Then $a$ is not a number; but rather, it is a vector (or line segment or interval) in $\mathbb{R}$. Now suppose $a$ and $b$ be two lengths in the plane $\mathbb{R}^2$. In modern notation we could write $a$ and $b$ in vector notation (with some loss of generality) as $a = [\alpha 0]$ and $b = [0 \beta]$. In other words, $a$ is a vector on the $x$–axis and $b$ is a vector on the $y$–axis: see figure **??**. Now we are in the position to define the product $ab$ as the *fundamental parallelogram* of the vector $a + b = [\alpha\beta]$, i.e the square whose vertices have coordinates $(0,0), (\alpha,0), (0,\beta), (\alpha,\beta)$. Using

[6] Ancient Greek unit; one orgyia $\approx$ 1.85m

[7] For further details on Descartes' contribution to the abstraction of math: see [Och15]

sophisticated notation this procedure can be expressed as a map

$$\Lambda(\mathbb{R}^2)^1 \oplus \Lambda(\mathbb{R}^2)^1 \to \Lambda(\mathbb{R}^2)$$
$$(a,b) \mapsto a \wedge b$$

where $\Lambda(\mathbb{R}^2)$ is called the *exterior algebra of* $\mathbb{R}^2$, $\Lambda^1$ is essentially the set of one–dimensional vectors (lengths) and $a \wedge b$ is the "wedge–product" of the vectors $a$ and $b$, which returns the parallelogram defined by $a + b$. These are well–defined objects and operations in modern geometry, and the wedge–product is essentially a form of "addition" in the exterior algebra. This modern formalism not only reflects the notion of taking products of lengths known to the Greeks and early Europeans, but it also hints towards why pre–$17^{th}$ century mathematicians were inclined to view the product of lengths as an addition operation. Indeed our construction of $a \wedge b$ involved taking the four "fundamental points" defined by $a + b = \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$. With this example in mind it is not entirely fair to cast judgment on the ancients for being too naïve to realize a unification of the operations on numbers and magnitudes. Their viewpoint that magnitudes carried additional information other than an abstract quantity is simultaneously narrow-minded and incredibly foresightful. Narrow-minded since the prototypical early mathematician *only* viewed magnitudes in this fashion, and this agrees with Bos's analysis. But these mathematicians' work is also surprisingly sophisticated once you work out all of the formal details; in particular since much of modern mathematics deals with studying a given object *that is equipped with additional structure*. Early mathematicians simply lacked adequate systems of symbolism and abstraction to realize and understand the high–level mathematical structure that they were studying. Hence Bos's original claim that mathematicians failed to recognize that the operations on numbers, magnitudes, and ratios could be reconciled under a common framework is only part of the picture. The operations amongst numbers, magnitudes, and ratios may obey the same rules, but the global mathematical system of $\mathbb{N}$ is different from $\mathbb{R}$ or $\mathbb{Q}$. An interesting open problem in mathematics is whether there exists an *implicit* definition of $\mathbb{Z}$[8] inside of $\mathbb{Q}$, meaning given a definition of $\mathbb{Q}$ that is independent of $\mathbb{Z}$, can we naturally recover $\mathbb{Z}$ as the subset $\{\cdots, \frac{-2}{1}, \frac{-1}{1}, \frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \cdots\}$ of $\mathbb{Q}$? The fact of the matter is that there are many ways of realizing $\mathbb{Z}$ as a subset of $\mathbb{Q}$, all of which are algebraically isomorphic[9]. Since any implicit definition of $\mathbb{N}$ in $\mathbb{Q}$ would imply one of $\mathbb{Z}$ in $\mathbb{Q}$, to say that $\mathbb{N}$ and $\mathbb{Q}^+$ (i.e numbers and positive rational ratios) are close to being isomorphic only tells part of the story

[8] The integers $\{\cdots, -2, -1, 0, 1, 2, 3, \cdots\}$
[9] as rings

of mathematics, and that their study as fundamentally distinct mathematic objects is a necessary consequence of an abstract axiomatic approach to maths. Of course, these tools were not available to mathematicians until the $17^{th}$ century[10].

2. In this second example, we imagine that a magnitude $a$ is not simply the length of an interval, the area of a rectangle, or the weight of a physical object; but rather, the value $a$ represents *both* the physical interval (resp. rectangle, bullion) *and* its length (resp. area, weight). In other words, $a$ is an element of the set

$$\{\mathcal{R} = \{(r, \alpha) \in R \times \mathbb{R} : r \text{ is a rectangle and } \alpha \text{ is the area of } r\}$$

where $R$ is set of all rectangles and $\mathbb{R}$ is the set of areas. Note that the space $R \times \mathbb{R}$ is 2–dimensional (one dimension of rectangles plus one dimension of numbers). Using Bos's notation, let $r(a, b) = \text{rect}(a, b) \in R$ denote the rectangle with side lengths $a$ and $b$. The corresponding element in $\mathcal{R}$ is the tuple $(r(a, b), ab)$, where $ab$ is the usual product of real numbers. The observation that $r(a + c, b) = r(a, b) + r(c, b)$, included in the Greek's "geometrical algebra," relfects the distributive property of real numbers. At the level of the space $\mathcal{R}$, this property reads $(r(a+c, b), (a+c)b) = (r(a, b), ab)+(r(c, b), cb)$. The failure of mathematicians to recognize that $R$ is "isomorphic" to $\mathbb{R}$ can be now reinterpreted as a failure to recognize that $\mathcal{R}$ is a one dimensional subspace of $R \times \mathbb{R}$. This is analogous to how the diagonal line $y = x$ in the plane $\mathbb{R}^2$ can be regarded as a single copy of the real line $\mathbb{R}$. This example demonstrates that early mathematicians were making their efforts under the guise of some sophisticated mathematics, which is a problem since they were unaware of this fact. In fact, only with Descartes' coordinate geometry and Leibniz's introduction of symbolism could this problem be solved.

Under the framework of ratios of magnitudes, Bos notes that it is possible to define a closed operation of addition if one interprets $\frac{ad+bc}{bd}$ as "the ratio of line segment lengths that is canonically geometrically deduced from the ratio of areas $ad + bc : bd$". History shows that his idea was not embraced, and Bos observes that up until the $19^{th}$ century "ratios were not considered primary objects of mathematics, but instead were relations between objects." Perhaps it was this lower–tier status of ratios that obstructed the Greeks from developing a "universal system of mathematics." On a different note, historians of science and mathematicians are in common agreement that mathematicians throughout history have been drawn to *mathematical elegance*, which may be loosely described as a special simplicity of certain mathematical facts. In comparison to the elegance of the Greek's proof that $\sqrt{2}$ is irrational or that the set of prime numbers is

[10]Really one could argue the full force of these tools were not available until the post–Bourbaki era in the $20^{th}$ century

infinite [c.f Appendix A], the idea of having to temporarily change dimensions in order to define the addition of ratios (e.g $\{$1–d lines $a, b, c, d\} \rightarrow \{$2–d rectangles $ad, bc, bd\} \rightarrow \{$1–d lines $ad+bc, ad\}$) is non–elegant. We believe that this relative lack of simplicity was an additional barrier for the early adoption of reconciliatory ratio arithmetic. We reserve precisely defining *mathematical elegance* since our belief necessarily implies that mathematicians' understanding of elegance evolves over time. This is possibly a discussion for a future paper.

In summary, Bos's arguments provide a good framework for historians of science to understand exactly how mathematics was impacted by Descartes' coordinate geometry, as well as the algebra and calculus revolutions that followed with figures such as Newton and Leibniz; however, Bos's main argument could easily be misinterpreted or misunderstood. Therefore, it is important that we revisit the discussion with a slightly more modern mathematical veiwpoint so that the fundamental structure of Greek/proto–European mathematics and the reasoning of Greek/proto–European mathematicians can be better understood.

# A Appendix

**Theorem.** *$\sqrt{2}$ is irrational.*

*Proof.* Suppose towards a contradiction that $\sqrt{2}$ is rational and, when written in lowest terms, is equal to $\frac{p}{q}$ for some integers $p$ and $q$. Then

$$2 = \frac{p^2}{q^2} \tag{3}$$

so $2q^2 = p^2$. If $p$ were odd, then $p^2$ would be odd; but $2q^2$ is clearly even, so $p$ is even. Thus there exists $k \in \mathbb{Z}$ so that $p = 2k$. Substituting $p = 2k$ into 3 and rewriting the equation yields $q^2 = 2k^2$; which, by the same argument as before, implies that $q$ is equal to $2j$ for some $j \in \mathbb{Z}$. It follows that $\frac{p}{q} = \frac{2k}{2j}$ is not written in lowest terms, so we arrive at a contradiction. We conclude that $\sqrt{2}$ is not equal to any rational number. □

**Theorem.** *There are infinitely many prime numbers.*

*Proof.* Suppose towards a contradiction there are finitely many primes, labelled $p_1, p_2, \ldots, p_N$ and consider the number $X = p_1 p_2 \cdots p_N + 1$. Observe that $X$ divided by any one of the $p_i$ returns a remainder of 1, so $X$ is not divisible by any of the $p_i$. This means that $X$ itself is a prime number that does not appear in the set $\{p_1, \ldots, p_N\}$, which contradicts the assumption that there are finitely many primes. □

# References


[Acz06] A. D. Aczel. *The Artist and the Mathematician.* Thunder's mouth press, 2006.

[Ati88] Atiyeh, G. N. and Oweiss, I. M. *Arab Civilization: Challenges and Responses Studies in Honor of Dr. Constantine Zurayk.* SUNY press, 1988.

[Bos84] H. J. M. Bos. Arguments on motivation in the rise and decline of a mathematical theory; the "construction of equations", 1637 - ca. 1750. *Archive for History of Exact Sciences*, 30(3/4):331–380, 1984.

[Bos90] Henk J.M. Bos. Numbers, Magnitudes, and Ratios. *Mannheim (Wissenschaftsverlag)*, 1990.

[Boy91] C. B. Boyer. *A History of Mathematics.* Wiley, 1991.

[Bro91] Gary I. Brown. The evolution of the term "mixed mathematics". *Journal of the History of Ideas*, 52(1):81–102, 1991.

[Bro15] Gary I. Brown. The establishment of 'mixed mathematics' and its decline 16001800. *Academy of Science*, 2015.

[CJ90] A. Cunningham and N. Jardine. *Romanticism and Science*. Cambridge Uni Press, 1990.

[EKSS13] M. Epple, T. H. Kjeldsen, and R. Siegmund-Schultze. From "mixed" to "applied" mathematics: tracing and important dimension of mathematics and its history. *Oberwolfach Report*, 2013.

[GK07] Steven G. Krantz. The history and concept of mathematical proof. 01 2007.

[Goo91] D. Goodman. *The Rise of Scientific Europe*. Hodder and Stoughton, 1991.

[Gro67] A. Grothendieck. Elements de geometrie algebrique. *Publications Mathematiques*, (4-32), 1960-1967.

[Gui90] Niccolo Guicciardini. The development of newtonian calculus in britain, 1700-1800. *Science*, 250(4977):144–144, 1990.

[Gui04] Niccol Guicciardini. Dot-age: Newton's mathematical legacy in the eighteenth century. *Early Science and Medicine*, 9(3):218–256, 2004.

[Mul16] S. Muller-Stach. Richard Dedekind: Style and Influence. *ArXiv e-prints*, December 2016.

[Och15] Tynan Ochse. Rationalism and the Break from Greek Tradition: The Abstraction of Mathematics in the 17th Century. 2015.

[Och17] Tynan Ochse. Leibnizian Mathematics (Calculus). Slides presentation given in the class HPS 157: Mathematization of 17 and 18th c. sciences, 2017.

[Pie98] James Pierpont. Early history of galois' theory of equations. *Bull. Amer. Math. Soc.*, 4(7):332–340, 04 1898.

[Ras94] Rashed, R. and Armstrong, A. *The Development of Arabic Mathematics*, volume 156 of *Boston Studies in the Philosophy and History of Science*. Springer, 1994.

[Ras14] Rashed, R. *Classical Mathematics from al-Kwharizmi to Descartes*. Culture and Civilization in the Middle East. Routledge, 2014.