Foundations of Arithmetic Floer (Co)homology

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March 14, 2018

1 Introduction

The purpose of this article is to lay the foundation for a *p–adic Floer cohomology theory*. The meaning of this terminology is encapsulated by the series of results and conjectures that follow. Roughly speaking, the author views p–adic Floer theory as a device to better understand arithmetic invariants coming from certain classes of fiber products (e.g Selmer groups) that naturally arise in Iwasawa theory and p–adic Hodge theory. In a different vein, we also expect p–adic methods to also be useful when applied to classical Floer homology theories, especially knot Floer homology: see conjecture 5. Minhyong Kim's recent paper on *Arithmetic Gauge Theory* [14] includes a promising route towards the former. For brevity and coherence with our discussion of classical Floer theories, we summarize section 11 of Kim's paper below: see also section 2.1.

Let K be a number field with ring of integers $\mathcal{O}_K$, and suppose R is a sheaf on $X := \operatorname{Spec}\mathcal{O}_K$. For a closed point $v \in X$ corresponding to the prime $\mathfrak{p}_v \subset \mathcal{O}_K$, define $\mathcal{O}_v = \varprojlim \mathcal{O}_K/\mathfrak{p}_v^n$, $K_v = \operatorname{Frac}(\mathcal{O}_v)$, $\mathfrak{X}_v = \operatorname{Spec}\mathcal{O}_v$, and $\mathfrak{K}_v = \operatorname{Spec} K_v$. Let S be a finite collection of closed points in X and let X^S denote the scheme $X \backslash S = \operatorname{Spec}\mathcal{O}_K[S^{-1}]$. Fix an algebraic closure $\overline{K}$ (resp. $\overline{K_v}$) of K (resp. K_v) and let π_S (resp. π_v) denote $\pi_1^{\acute{e}t}(X^S)$ (resp. $\operatorname{Gal}(\overline{K_v}/K)$). Consider the moduli spaces $\mathcal{M}(X^S, R)$, $\mathcal{M}(\mathfrak{K}_v, R)$, and $\mathcal{M}(\mathfrak{X}_v, R)$ of *principal R–bundles* over X^S, $\mathfrak{K}_v$, and $\mathfrak{X}_v$, respectively, together with the natural restriction maps

$$\mathcal{M}(X^v, R) \xrightarrow{\operatorname{loc}_S} \textstyle\prod_{v\in S}\mathcal{M}(\mathfrak{K}_v, R) \xleftarrow{r_S} \textstyle\prod_{v\in S}\mathcal{M}(\mathfrak{X}_v, R) \tag{1}$$

And define

$$C(X, S; R) := \mathcal{M}(X^S, R) \times_{\prod_v \mathcal{M}(\mathfrak{K}_v, R)} \prod_v \mathcal{M}(\mathfrak{X}_v, R)$$

If R is a sheaf of unipotent $\mathbb{Q}_p$–algebraic groups, then $\mathcal{M}(X^S, R) \cong H^1_{cts}(\pi_S, R)$, $\mathcal{M}(\mathfrak{K}_v, R) \cong H^1_{\eta_{\acute{e}t}}(\pi_v, R)$, and

$$\mathcal{M}(\mathfrak{X}_v, R) \cong \begin{cases} H^1_{cts}(\pi_v/I_v, R^{I_v}) & v \nmid p \\ H^1_f(\pi_v, \operatorname{Crys}_v(R)) & v \mid p \end{cases}$$

Here H^*_{cts} denotes conintuous cohomology, $H^*_{\eta_{\acute{e}t}}$ is local cohomology, H^*_f is unramified cohomology, I_v is the inertia subgroup of π_v, and $\operatorname{Crys}_v(R)$ is the maximal π_v–equivariant subgroup of R that is Crystalline[1]. **We drop these subscripts in what follows**. Similar identifications of $\mathcal{M}(-, R)$ with moduli spaces of Galois representations hold when R is a subquotient of $GL_n(E)$ for some finite extension of $\mathbb{Q}_p$.

In classical Floer theory, R is typically a complex semisimple Lie group (viewed as a constant sheaf), and the $\mathcal{M}(-, R)$ are replaced by the moduli spaces of principal R–bundles *with connection* over specially defined submanifolds of an ambient 3–manifold. Under these hypotheses, one realizes the images of loc_S and r_S as Lagrangian submanifolds of a symplectic manifold: see section 2 for details. Classically, cotangent bundles are one natural source of symplectic manifolds, and the critical loci of real–valued functions provide Lagrangian submanifolds. These facts serve to motivate the following Theorem.

[1] As Kim notes, it is possible to enhance our definitions here using the tools of p–adic Hodge Theory

Theorem 1 (Kim). *Suppose S contains all places dividing a fixed prime $p \in \mathbb{Z}$, and assume R is either a sheaf of unipotent $\mathbb{Q}_p$–analytic groups with a continuous $\pi_1(X^S)$–action or a subquotient of $GL_n(E)$ for some finite extension E of $\mathbb{Q}_p$. Let $T^*(1)R \coloneqq L^*(1) \rtimes R$ denote the (Tate) twisted cotangent bundle, where L is the Lie algebra of R. Then,*

1. *For any closed point $v \in X$, the tangent space*

$$T_{\widetilde{c_v}} H^1(\pi_v, T^*(1)R) \cong H^1(\pi_v, L(c_v))^* \times H^1(\pi_v, L(c_v))$$

has the structure of a symplectic vector space with symplectic form ω_v given by

$$\omega_v((f,c),(f',c')) = f \cdot c' - f' \cdot c$$

where $\cdot$ is the inner product structure on $T_{\widetilde{c_v}}$, $\widetilde{c_v}$ is a cocylce of π_v with values in $T^*(1)R$, c_v is its image in R, and $L(c_v)$ is L with the π_v-action twisted by the adjoint action of c_v. In particular, summing over $v \in S$ gives a symplectic structure on

$$\prod_{v\in S} \left[H^1(\pi_v, L(c_v))^* \times H^1(\pi_v, L(c_v))\right]$$

Hence, under a suitable geometrization of the underlying spaces,

$$\prod_{v\in S} H^1(\pi_v, T^*(1)R)$$

has the structure of a symplectic p–adic analytic variety. Moreover, if $\widetilde{c_v}$ is crystalline or unramified for each $v \in S$, then

$$\prod_{\substack{v\in S\\ v\nmid p}} \left[H^1(\pi_v, L(c_v)^*(1)) \times H^1(\pi_v, L(c_v))\right] \times \prod_{\substack{v\in S\\ v\mid p}} \left[H^1(\pi_v/I_v, (L(c_v)^*(1))^{I_v}) \times H^1(\pi_v/I_v, (L(c_v))^{I_v})\right]$$

is a Lagrangian subvariety of $\prod_{v\in S}\left[H^1(\pi_v, L(c_v)^*(1)) \times H^1(\pi_v, L(c_v))\right]$

2. *The image of $H^1(\pi_S, T^*(1)R)$ under the locatlisation map*

$$\mathrm{loc}_S : H^1(\pi_S, T^*(1)R) \to \prod_{v\in S} H^1(\pi_v, T^*(1)R)$$

is a Lagrangian subvariety. In particular, the intersection

$$C(X,S;T^*(1)R) = \mathcal{M}(X^S, T^*(1)R) \times_{\prod_v \mathcal{M}(\mathfrak{K}_v, T^*(1)R)} \prod_v \mathcal{M}(\mathfrak{X}_v, T^*(1)R)$$

can be regarded as an "arithmetic Lagrangian intersection."

We take as fact that the natural projection map $H^1(\pi_S, T^*(1)R) \to H^1(\pi_S, R)$ is split. Therefore, $C(X,S;T^*(1)R)$ admits a split projection to

$$C(X,S;R) \coloneqq \mathcal{M}(X^S, R) \times_{\prod_v \mathcal{M}(\mathfrak{K}_v, R)} \prod_v \mathcal{M}(\mathfrak{X}_v, R)$$

which should also be regarded as a $\mathbb{Q}_p$–analytic Lagrangian intersection.

Conjecture 1. *There exists canonical étale perverse sheaves of vanishing cycles $\mathscr{P}^\bullet_{T^*(1)R}$ and $\mathscr{P}^\bullet_R$ on $C(X,S;T^*(1)R)$ and $C(X,S;R)$, respectively. In particular, the hypercohomology groups $\mathbb{H}(\mathscr{P}^\bullet)$ give an "arithmetic Floer–cohomological" invariant of $\prod_{v\in S} \mathcal{M}(\mathfrak{K}_v, R)$.*

We expect Theorem 1 to hold more generally if we replace X with an arbitrary variety over K. This next example illustrates the significance of the intersections $C(X, S; R)$ in arithmetic geometry (a lecture series at AWS 2018 was devoted to their study), and is of particular interest to the author.

Example 1. Let A denote an elliptic curve over K with good reduction at all primes $v \mid p$. and let R denote the Lisse sheaf associated to $V_p := T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where $T_p(A)$ is the Tate module. Suppose S contains $\{v : v \mid p\} \cup \{\mathfrak{p} : A \text{ has bad reduction at } \mathfrak{p}\}$. If $v \in S \backslash \{v : v \mid p\}$, then

$$H^1_f(\pi_v, V_p) = H^1(\pi_v/I_v, V_p^{I_v})$$

If $v \mid p$, then $H^1_f(\pi_v, V_p) \subset H^1(\pi_v, V_p)$ is the subspace of crystalline torsors. Then the fiber product in diagram 1 is the p^∞–Selmer group of A tensored with $\mathbb{Q}_p$:

$$Sel(A, \mathbb{Q}_p) = H^1(\pi_S, V_p) \times_{\prod_{v \in S} H^1(\pi_v, V_p)} \prod_{v \in S} H^1_f(\pi_v, V_p)$$

The finiteness conjecture for the p^∞–part of the Tate-Shafarevich group of A over F can be reformulated, using the help of a certain exact sequence [25, Page 6], as follows

Conjecture 2. $\mathrm{corank}_{\mathbb{Z}_p} Sel(A, \mathbb{Z}_p) = \dim_{\mathbb{Q}_p} Sel(A, \mathbb{Q}_p) = \mathrm{rank}_{\mathbb{Z}} A(K)$

By definition, $\mathrm{corank}_{\mathbb{Z}_p} Sel(A, \mathbb{Z}_p)$ is the $\mathbb{Z}_p$–rank of the Pontryagin dual $Hom_{cts}(Sel(A, \mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$. This latter space is interesting to the author from a related point of view: see remark 3. Specifically, $Hom_{cts}(\mathbb{Q}, \mu_{p^\infty})$ should be identifiable with the p–adic solenoid. This is one justification for our study of solenoids. The remainder of this section describes a potentially novel route towards understanding relationships between Selmer groups, elliptic curves, and their L–functions. It also gives additional motivation for our study of adelic and p–adic solenoids. We begin by recalling a Theorem of Étienne Ghys on the structure of the unit tangent bundle of the modular surface.

Theorem 2 (Ghys)**.** *The unit tangent bundle of the modular surface can be identified with the complement of the trefoil knot in S^3. Under this identification, closed geodesics on $\mathbb{H}^2/PSL_2(\mathbb{Z})$ lift to knots in $S^3/T(2,3)$.*

Definition 1. A *modular knot* is a knot which arises as the lift of some closed geodesic on $\mathbb{H}^2/PSL_2(\mathbb{Z})$ to its unit tangent bundle.

Let K be a knot in S^3 with Alexander polynomial $\Delta_K(t)$ and let $\pi_K = \pi_1(S^3 \backslash K)$ denote the fundmental group of the knot complement. It is easy to show that $\pi_K^{ab} = \pi_K/[\pi_K, \pi_K] \cong \mathbb{Z}$. For each $z \in \mathbb{C}$, ψ_z denotes the homomorphism $\psi_z : \pi_K \to \mathbb{C}$ that sends a generator τ of π_K^{ab} to z. The ψ_z define a linear system $V(z)$ over $X_K := S^3 \backslash K$. Mazur [16] observed that

$$\dim_{\mathbb{C}} H_1(X, V(z)) = \text{Order of vanishing of } \Delta_K \text{ at } z \tag{2}$$

Now let $\Sigma = \lim\limits_{\leftarrow_i} (S^1, m_i)$ be a one–dimensional solenoid; that is, an inverse limit of circles with transition maps $S^1 \xrightarrow{[m_i]} S^1$ define by $z \mapsto z^{m_i}$. Without loss of generality we may assume that each m_i is a prime number [6]. If Σ comes with an embedding into S^3, then we can view Σ as an infinite cable knot, i.e the result of cabling the copy of S^1 at the $(i+1)^{th}$-layer, viewed as a knot K_{i+1}, around K_i with framing $a_i \in \mathbb{Z}$. Let Σ_n denote the cable knot at the n^{th}-layer in the construction of Σ. It is well-known that the Alexander polynomials of a cable knot is given by the formula

$$\Delta_{\Sigma_n} = \prod_{i=0}^{n} \Delta_{K_i}(t^{a_i})$$

In particular, we can formally define

$$\Delta_\Sigma(t) := \prod_{i=0}^{\infty} \Delta_{K_i}(t^{a_i}) \tag{3}$$

For each i let $X_{i,\infty}$ denote the infinite cyclic cover of $X_i \coloneqq S^3\backslash K_i$. $X_{i,\infty}$ has deck group $\mathrm{Gal}(X_{i,\infty}/X_i)$ isomorphic to $\mathbb{Z}$ generated by τ_i, hence $H_1(X_{i,\infty})$ inherits a natural action by $\mathbb{Z}[\mathrm{Gal}(X_{i,\infty}/X_i)] \cong \mathbb{Z}[t^{\pm 1}]$. Then, up to multiplication by a unit in $\mathbb{Z}[\mathrm{Gal}(X_{i,\infty}/X_i)]$, one has

$$\Delta_{K_i}(t) = \det(t \cdot id - \tau_i \mid H_1(X_{i,\infty}))$$

Substituting the right-hand side into equation 3, one has

$$\Delta_\Sigma(t) = \prod_{i=0}^{\infty} \det(t^{a_i} \cdot id - \tau_i \mid H_1(X_{i,\infty})) \tag{4}$$

Since $S^3\backslash\Sigma = \varinjlim S^3\backslash K_i$ and $\pi_1(S^3\backslash\Sigma) = \varinjlim \pi_1(S^3\backslash K_i)$ [6], we obtain a well defined linear system $\mathcal{V}(z) = \varinjlim V_i(z)$ over $S^3\backslash\Sigma$, where $V_i(z)$ is the linear system over $S^3\backslash K_i$ in equation 2. Note that we cannot expect an analogue of 2 in terms of singular homology since solenoids are not locally connected. On the upshot, Čech homology behaves well for pathological spaces.

Conjecture 3. $\dim_{\mathbb{C}} \check{H}_1(S^3\backslash\Sigma, \mathcal{V}(z)) =$ *Order of vanishing of* Δ_Σ *at* z

Alexander duality gives the isomorphism

$$\check{H}_1(S^3\backslash\Sigma, \mathcal{V}(z)) \cong \check{H}^1(\Sigma, \mathcal{V}(z))$$

So we can phrase conjecture 3 entirely in cohomological terms. The resemblance of solenoidal Alexander polynomials to Euler products of L–functions of elliptic curves (constructed via Iwasawa modules) is immensely striking: see also [25] and [18, Chapter 11]. However, this formulaic similarity is not enough to suspect a deep connection between knots and elliptic curves. Theorem 2 provides one link between the two worlds. Another connection that we shall discuss at a later time, and which naturally ties in the study of solenoids, is provided via Iwasawa Theory and the work of Scholze and Kucharzyk: see remark 3 and section 3.

Conjecture 4. *Let p be a prime and let A and elliptic curve over $\mathbb{Q}$ with good reduction at p. Then there exists a p–adic solenoid Σ_p (i.e $m_i \in p\mathbb{Z}$ for all i) such that*

1. *For each i, the knot K_i at the i^{th} layer of Σ_p is a modular knot.*
2. $\dim_{\mathbb{Q}_p} Sel(A, \mathbb{Q}_p) = \dim_{\mathbb{C}} \check{H}_1(S^3\backslash\Sigma_p, \mathcal{V}(1))$

Combining conjectures 3 and 4 with the Bloch–Kato and BSD conjectures, one obtains the conjectural string of equalities

$$\mathfrak{o}(L(A,s),1) = \mathrm{rank}_{\mathbb{Z}}(A) = \dim_{\mathbb{Q}_p} Sel(A, \mathbb{Q}_p) = \dim_{\mathbb{C}} \check{H}^1(\Sigma, \mathcal{V}(1)) = \mathfrak{o}(\Delta_\Sigma(t), 1)$$

where $\mathfrak{o}(f(x), y)$ denotes the order of vanishing of f at $x = y$. One route towards investigating the right "half" of the above equalities is to engage in a more thorough study of arithmetic properties of knots[2]

Given a knot K, recall that for each prime p we can construct the *p–adic Alexander polynomail* of K as follows: For $p = \infty$ let X_∞ denote the infinite cyclic cover of $X_K = S^3\backslash K$ corresponding to the kernel of the map $\pi_K \to \mathbb{Z}$. Then $\Delta_K(t)$ is the characteristic polynomial of the meridian action on the Alexander module $H_1(X_\infty, \mathbb{Z}) \otimes \mathbb{Q}$. For finite primes p, define

$$H_1(X_{p^\infty}, \mathbb{Q}_p) \coloneqq H_1(X_{p^\infty}, \mathbb{Z}) \otimes \mathbb{Q}_p = \varprojlim_n H_1(X_{p^\infty}, \mathbb{Z}/p^n\mathbb{Z})$$

and define $\Delta_{K,[p]}(t)$ to be the characteristic polynomial of the meridian action on the *p–adic Alexander module* $H_1(X_{p^\infty}, \mathbb{Q}_p)$.

[2] Whatever that means

Conjecture 5. *Let K be a knot and $p \leq \infty$ a prime number. Then for each $n \in \mathbb{N}$ there exists a bigraded $\mathbb{Z}_p$-module*

$$\widehat{HFK}(K)_{[p^n]} = \bigoplus_{\mathfrak{m},\mathfrak{s}} \widehat{HFK}(K,\mathfrak{s})_{[p^n]}$$

such that

$$\Delta_{K,[p^n]}(t) = \sum_{\mathfrak{m},\mathfrak{s}} (-1)^{\mathfrak{m}} t^{\mathfrak{s}} \mathrm{rank}_{\mathbb{Z}_p}(\widehat{HFK}(K,\mathfrak{s})_{[p^n]})$$

is the reduction modulo p^n of the p–adic Alexander polynomial of K.

The gradeded structue in the above Theorem is dictated by the p–adic Maslov grading $\mathfrak{m}$ [28] and the "Iwasawa grading" $\mathfrak{s}$. Since p–adic Alexander polynomials are Iwasawa polynomials, their coeffecients are p–adic integers. When $p = \infty$, $\widehat{HFK}(K)_{[\infty]} = \widehat{HFK}(K)$ is the usual knot Floer homolgy, $\mathfrak{s}$ is the Alexander grading, and $\Delta_{K,[\infty]}(t) = \Delta_K(t)$ is the usual Alexander polynomial.

Some remarks...

Remark 1. It is plausible that one may formally define

$$\widehat{HFK}(K)_{[p^\infty]} = \bigoplus_{\mathfrak{m},\mathfrak{s}} \widehat{HFK}(K,\mathfrak{s})_{[p^\infty]}$$

where each $\widehat{HFK}(K,\mathfrak{s})_{[p^\infty]}$ is isomorphic to $\mathbb{Z}_p^{a_{\mathfrak{m},\mathfrak{s}}}$ for some $a_{\mathfrak{m},\mathfrak{s}} \in \mathbb{Z}_p$. While the interpretation of $\mathbb{Z}_p^{a_{\mathfrak{m},\mathfrak{s}}}$ is unclear, the author suspects a deep relationship with the theory of p–adic framed braids [12, 13], especially in the context of the solenoid analogue of Alexander's Theorem 6.

Remark 2. In principle, the first step towards an arithmetic analogue of Floer theory is to establish a relationship between arithmetic geometry and Morse theory. This MathOverflow post indicates that the algebraic analogue of Morse theory is the study of vanishing cycles and Lefschetz pencils, which agrees with our attempt to identify an étale sheaf of vanishing cycles on ultrametric Lagrangian intersections.

Taking inspiration from Witten's deformation theory [5], we should be able to perturb some p–adic dg category, using p–adic Hodge theory, into a "Morse category" that admits an A_∞ structure in the sense of Fukaya [20]. In arithmetic geometry one frequently encounters infinite dimensional spaces, so a Floer–homological veiwpoint would be natural in this framework.

Whilsts attempting the constructions described in the sections that follow, it appears necessary to exploit the p–adic microlocal deformation theory of [10, 7, 9] as well as the relationship between formal group laws and complex cobordisms, originally noted by Quillen [23]. Additionally, Witten's deformation DR $\rightsquigarrow$ Morse of a complex deRahm complex into a Morse complex [5] reminds the author of Tsuji's construction of the syntomic complex Syn from the (log) deRahm complex DR using the theory of Filtered Isocrystals [27]. The syntomic site on a scheme X is generated by flat, locally complete intersections over X. When X is smooth over a Henselian discrete valuation ring, then the constant sheaf shifted by $\dim X + 1$ is an étale perverse sheaf. This gives one of the first hints towards a solution to conjecture 1.

Remark 3. Choose the generic point $\mathfrak{K}_v(\zeta_{p^\infty}) := \mathrm{Spec}\, K_v(\zeta_{p^\infty}) \to X^S[\zeta_{p^\infty}]$. If $\mathrm{Gal}(\overline{K_v}/K_v)$ is pro-p (which occurs whenever $v \mid p$), then the work of Kucharczyk and Scholze [15, Theorem 1.11] shows there exists a compact Hausdorff space Y_{p,K_v} such that

$$\pi_1^{\acute{e}t}(\mathfrak{K}_v(\zeta_{p^\infty})) \cong \mathrm{Gal}(\overline{K_v}/K_v(\zeta_{p^\infty}))$$

and

$$H^i(Y_{p,K_v}, \mathbb{Z}/p^m\mathbb{Z}) \cong H^i(\mathrm{Gal}(\overline{K_v}/K_v(\zeta_{p^\infty})), \mathbb{Z}/p^m\mathbb{Z})$$

Moreover, there exists compact Hausdorff spaces X_K and X_{K_v} such that the the functor taking a finite extension E (resp. E_v) of K (resp. K_v) induces an equivalence of categories

$$\{\text{finite extensions of } K \text{ (resp. } K_v)\} \equiv \{\text{connected finite covers of } X_K \text{ (resp. } X_{K_v})\}$$

and so $\mathrm{Gal}(\overline{K}/K) \cong \pi_1^{\acute{e}t}(X_K)$ (resp. $\mathrm{Gal}(\overline{K_v}/K_v) \cong \pi_1^{\acute{e}t}(X_{K_v})$) [15, Theorem 1.5]. When K is algebraically closed, X_K can be identified with the adelic solenoid $\mathbb{A}/\mathbb{Q}$. This further motivates our study of p–adic solenoids, while also suggesting that one may be able to partially reconcile classical Guage theory and Kim's Arithmetic Guage theory under a common topological framework.

Remark 4. As pointed out to me by Kenji Fukaya via email, in order for the notion of p–adic Lagrangian cohomology to make sense, one first needs to generalize the definition of Lagrangian submanifolds to the p–adic world. Indeed, Lagrangian submanifolds of an ambient symplectic manifold locally look like the submanifold $\mathbb{R}^n$ of $\mathbb{R}^{2n}$ [3]. We will make a precise geometric definition at a later time when the situation is more clear to the author, though the fundamental result is Ostrowski's Theorem, which states that every absolute value on $\mathbb{Q}$ is equivalent to either the standard Euclidean absolute value or a p–adic one. So the possible Cauchy–completions of $\mathbb{Q}$ are exactly $\mathbb{R}$ and $\{\mathbb{Q}_p\}_{p:\text{ prime}}$. Since $\mathbb{R}$ and $\mathbb{Q}_p$ are isomorphic as sets, we can view $\mathbb{Q}_p$ as the set $\mathbb{R}$ endowed with an exotic metric. The same philosophy applies to $\mathbb{C}$ and $\mathbb{C}_p$.

The structure of this paper is as follows: in section 2 we recall the classical approach to defining Floer cohomological invariants in low-dimensional topology. In section 2.1 we summarize the justification for Theorem 1. Section 2.1 begins an informal discussion on p–adic Floer–type *homological* invariants. Finally, in section 3 we begin working towards further evidence for conjecture 4.

Some notation

$\mathbb{Q}$ denotes the rational numbers, $\mathbb{Q}_p$ is the Cauchy completion of $\mathbb{Q}$ with respect to the p–adic absolute value $|-|_p$ ($\mathbb{Q}_\infty = \mathbb{R}$ and $|-|_\infty$ is the usual Archimedean metric), and $\mathbb{C}_p$ is the p–adic completion of any fixed algebraic closure $\overline{\mathbb{Q}_p}$ ($\mathbb{C}_\infty = \mathbb{C}$).

Gp, **Ab**, **Rng**, and **Sch** denote the category of groups, abelian groups, rings, and schemes respectively.

Given a topological space X, **Top**(X), **Ét**(X), **Zar**(X) denote the topological, étale, and Zariski sites of X, respectively. **Sh**(X) and $\mathbf{Sh}^\nabla$ indicate the category of sheaves on X and the category of *sheaves* with connection on X[4].

2 Floer (co)homology

We first recall the general procedure for constructing Floer Homology theories. Since most Floer theories are either known to or conjectured to arise as special cases of Lagrangian Floer homology, this version is what we describe here.

Definition 2. Let $\mathcal{M}$ be a manifold over $\mathbb{R}$ of dimension $2n$. An *almost complex structure* J on $\mathcal{M}$ is a choice of complex structure at each tangent space $T_m\mathcal{M}$.

Suppose $\mathcal{M}$ is a symplectic manifold with an almost complex structure J, and assume there exists transversal Lagrangian submanifolds L_1 and L_2 in $\mathcal{M}$, so $C(L_1, L_2) := L_1 \cap L_2$ is 0–dimensional. Let

[3] This fact is stated without proof in this nLab page. However, the author is skeptical since the definitions of a symplectic manifold and Lagrangian submanifolds provided in every source consulted so far seem to indicate that the definitions should work over any base field, provided that we replace the word "manifold" with "analytic space" or "rigid–analytic space".

[4] Sheaves with connection are automatically locally free.

$\mathcal{C}(L_1, L_2)$ be the free $\mathbb{Z}$–module generated elements of $C(L_1, L_2)$. When $\mathcal{M}$ is compact, $\mathcal{C}(L_1, L_2)$ is finitely generated. For general $\mathcal{M}$, $C(L_1, L_2)$ is called a *d–critical locus*, which for our purposes can be simply understood as a divisor of some derived scheme–theoretic object. Specifically,

Definition 3. Let $\mathcal{M} \subset \mathbb{A}^n_K$ be a smooth affine variety over a field K (usually $\mathbb{R}$ or $\mathbb{C}$), $f : \mathcal{M} \to \mathbb{C}$ a regular function, and define a section $\Gamma_{df} : M \to \Omega_M$ by $x \mapsto (x, df(x))$. The *derived critical locus* of f, denoted $\mathcal{D}\mathrm{Crit}(f)$, is the derived fiber product

$$\begin{array}{ccc} \mathcal{D}\mathrm{Crit}(f) & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow 0 \\ \mathcal{M} & \xrightarrow{\Gamma_{df}} & \Omega_M \end{array}$$

in $\mathcal{D}\mathbf{Coh}(\mathcal{M})$; and is given by the differential graded scheme $(\mathcal{M}, \mathcal{D}^\bullet, \mathfrak{d}f)$, where the underlying graded ring structure is $\mathscr{D}^{-i} = \bigwedge^i TM$, and the differential map $\mathfrak{d}f$ is contraction (interior product) by df.

Lagrangian Floer homology is defined as the homology of the complex $(\mathcal{C}(L_1, L_2), \partial)$, where the differential ∂ counts distinct pairs of gradient flow lines (ϕ_1, ϕ_2) connecting points in $C(L_1, L_2)$. Explicitly, the ϕ_i are realized as boundary arcs on embedded topological disks, and then ∂ takes the form

$$\partial x = \sum_{y \in C(L_1, L_2)} \sum^{*}_{\phi \in \pi_2(x,y)} |\mu(\phi)/\mathbb{R}| \cdot y \tag{5}$$

where $\pi_2(x, y)$ are the isotopy classes of Whitney disks with marked points x and y [c.f definition 6], the restricted sum Σ^* counts only those embeddings ϕ that satisfy some differential equation (e.g Cauchy–Riemann, Yang–Mills, Seiberg–Witten, etc), and $\mu(\phi)/\mathbb{R}$ is the moduli space of such *geometric* disks, modulo scaling[5]. In the Instanton, Heegaard, and Knot versions of Floer homology, the aforemorentioned differential equation can be deformed, in a precise sense, into the Cauchy–Riemann equation. In particular, Σ' counts *pseudoholomorphic* Whitney disks. More generally,

Definition 4. A *pseudoholomorphic* curve in an almost complex manifold X is a map $\phi : D \to X$ from a Riemann surface D with complex structure j that satisfies the Cauchy–Riemann differential equation:

$$J \circ \mathrm{d}\phi = \mathrm{d}\phi \circ j$$

This definition can be naîvely extended to a full subcategory of **Sch** as follows.

Definition 5. Let X be a scheme that admits a smooth model over $\mathbb{R}^{2n}$ and let J be the pullback of the canonical almost complex structure on $\mathbb{R}^{2n}$ to the tangent bundle of X^{an}, the analytification of $X(\mathbb{C})$, viewed as a scheme over $\mathbb{C}^n$. A *psuedoalgebraic* curve in X is a closed embedding $D \xrightarrow{\phi} X$ of a projective 1–dimensional scheme D such that $D^{an} \xrightarrow{\phi^{an}} X^{an}$ is a pseudo–holomorphic curve.

Definition 6. Let X be a manifold equipped with submanifolds L_1 and L_2 that intersect at points x and y. A *Whitney disk* in (X, L_1, L_2) is an embedding $\phi : D \to X$ of a real 2–dimensional disk with two marked points p and q such that $\phi(p) = x$, $\phi(q) = y$, $L_1 \cap \partial(\phi(D)) = \phi_1$, and $L_2 \cap \partial(\phi(D)) = \phi_2$, where ϕ_1 and ϕ_2 indicate the distinct closed hemispheres on ∂D whose endpoints are x and y. The moduli space of Whitney disks with marked points x and y is denoted $\pi_2(x, y)$ when the choice of L_1 and L_2 is unambiguous.

Now let X be a compact oriented 3–manifold, $R \in \mathbf{Sh}^\nabla(X)$, and suppose $X = X_1 \cup_\Sigma X_2$ is a Heegaard splitting of minimal genus. We also fix a basepoint $\zeta \in \Sigma$. Let $\mathcal{M}(-, R)$ (resp. $\mathcal{M}^\flat(-, R)$) denote the moduli space of R–bundles (resp. *flat* R–bundles) with connection, and consider the natural restriction maps

$$\mathcal{M}^*(X_1, R) \xrightarrow{r_1^*} \mathcal{M}^*(\Sigma, R) \xleftarrow{r_2^*} \mathcal{M}^*(X_2, R) \tag{6}$$

[5]To motivate the reduction mod scaling, recall that all Riemann surfaces of genus $g > 0$ admit a "pants decomposition" and the pair-of-pants factors are uniquely determined by the real–valued radius of the disk bounded by any one of the pant–legs

where $* \in \{\emptyset, \flat\}$. It is useful to alter the minimal Heegaard splitting by gluing a torus T^2 to Σ along a closed disk neighborhood of ζ to obtain a Heegaard Splitting $X = X_1(1) \cup_{\Sigma(1)} X_2(1)$ of higher genus. One then considers the moduli spaces of *twisted* $*$-R–connections, denoted $\mathcal{M}^*_{\rtimes}(-, R)$: see [1, §2] for definitions. In this situation there are also natural restriction maps:

$$\mathcal{M}^*_{\rtimes}(X_1(1), R) \xrightarrow{r^*_{\rtimes_1}} \mathcal{M}^*_{\rtimes^1}(\Sigma(1), R) \xleftarrow{r^*_{\rtimes_2}} \mathcal{M}^*_{\rtimes}(X_2(1), R) \tag{7}$$

We adopt the conventions $\mathcal{M}^*_{\rtimes^0}(Y, R) = \mathcal{M}^*(Y, R)$, $\mathcal{M}^*_{\rtimes^1}(Y, R) = \mathcal{M}^*_{\rtimes}(Y(1), R)$, and define

$$L^*_{\rtimes^j_i} := r^*_{\rtimes^j_i}\left(\mathcal{M}^*_{\rtimes^j}(X_i, R)\right)$$

for $i \in \{1, 2\}$, $j \in \{0, 1\}$, and $* \in \{\emptyset, \flat\}$. Set $C^*_{\rtimes^j}(X) := L^*_{\rtimes^j_1} \cap L^*_{\rtimes^j_2}$.

Proposition 3. *1. When R is algebraic and reductive, $C^\flat_{\rtimes}(X)$ is the representation variety* $\mathrm{Hom}(\pi_1(X), R)$.

2. If R is a complex semisimple Lie group (e.g $SU(2)$ or $SL_2(\mathbb{C})$), then $\mathcal{M}^*_{\rtimes^j}(\Sigma, R)$ is a symplectic manifold, the $L^*_{\rtimes^j_i}$ are Lagrangian submanifolds, and $C^*_{\rtimes^j}(X)$ is a d–critical locus.

Joyce [11] originally introduced the notion of d–critical loci, which provides a framework to understand complex Lagrangian intersections as a divisor of some derived geometric object. At least for now, we will not say any more about d–critical loci.

When $R = SU(2)$, the Atiya–Floer conjecture states that the *Instanton Floer Homology* of X is isomorphic to the Lagrangian Floer homology of the pair $\left(L^\flat_1, L^\flat_2\right)$. When $R = SL_2(\mathbb{C})$, Abouzaid and Manolescu identify a perverse sheaf of vanishing cycles $P^\bullet_{L^\flat_{\rtimes^j_1}, L^\flat_{\rtimes^j_2}}$, on $C^\flat_{\times^j}(X)$ [1], which is given by Bussi's general construction of perverse sheaves on complex Lagrangian intersections [4]. In particular, the hypercohomology groups

$$\mathbb{H}(P^\bullet_{L^\flat_{\rtimes^j_1}, L^\flat_{\rtimes^j_2}})$$

are $SL_2(\mathbb{C})$ Floer cohomological invariants of X. The non–compactness of $SL_2(\mathbb{C})$ is an obstruction to defining Lagrangian Floer homology directly via Gauge theory; nevertheless, it is conjectured that the dual of such a Floer homology would be isomorphic to $\mathbb{H}(P^\bullet_{L^\flat_{\rtimes^j_1}, L^\flat_{\rtimes^j_2}})$. By using different complex semisimple structure groups, the general technique of identifying perverse sheaves on Lagrangian intersections is anticipated to also provide cohomological dual groups to Heegaard Floer Homology and Knot Floer Homology.

2.1 Arithmetic Floer Cohomology

Keeping with the notation in the Introduction, in this section we ellaborate upon Theorem 1. The injective maps $\mathcal{O}_K \to K \to K_v$ and $\mathcal{O}_K \to \mathcal{O}_V[v^{-1}]$ induce open étale morphisms $S_v \to \operatorname{Spec} K \to X$ and $X^v \to X$. Similarly, the closed immersion $\operatorname{Spec} \kappa_v \to X$ and the natural inlusion $\mathcal{O}_K \to \mathcal{O}_v$ induce a closed étale morphism $\mathfrak{X}_v \to X$. All together one obtains the familiar open–closed (○–●) decomposition in the étale site:

$$X^v \xrightarrow{\circ} X \xleftarrow{\bullet} \mathfrak{X}_v$$

The precise formalism still needs to be worked out, but by appealing to faithfully flat (étale) descent and the above decomposition we write

$$X = X^v \cup_{\mathfrak{K}_v} \mathfrak{X}_v \tag{8}$$

One interprets the formal neighborhood $\mathfrak{X}_v$ as a "handlebody" around the prime v, while X^v is akin to a knot complement. let $R \in \mathbf{Sh}(X)$. Eventually one would like to choose $R \in \mathbf{Sh}^\nabla(\tilde{X})$, where $\tilde{X}$ is some étale or syntomic cover of X and the connection $\nabla : R \to \Omega^1_{\tilde{X}/X} \otimes_{\mathcal{O}_X} R$ is either the Gauss–Manin connection or some integral connection coming from a convergent filtered isocrystal over the Witt ring $W(\mathcal{O}_K/\mathfrak{p}_v)$, but for now we do not make any such assumption.

The standard method for defining Floer (co)homology is to first realize the relevant geometric spaces as Lagrangian submanifolds of some ambient symplectic manifold (e.g $L^\flat_{\rtimes_1^j}$ and $L^\flat_{\rtimes_2^j}$ in $\mathcal{M}^\flat_{\rtimes^j}(\Sigma, R)$ in section 2). When the Lagrangian intersection $C(X)$ is algebraic or complex-analytic, one can identify a perverse sheaf of vanishing cycles on $C(X)$ [4]. The hypercohomology of this complex gives the Floer cohomology of the symplectic manifold. In the arithmetic scenario, while Kim notes that "the underlying geometric foundation still needs to be worked out," it seems likely that one can identify an étale perverse sheaf of vanishing cycles $\mathscr{P}^\bullet_{X,S}$ on the arithmetic Lagrangian intersections described in the introduction and below, and then pass to the hypercohomology $\mathbb{H}(\mathscr{P}^\bullet_{X,S})$. The first steps towards this possibility would ideally involve appealing to known results; however, one may also need to develop a p–adic analogue for Bussi's construction.

Continuing on with the justification for Theorem 1, suppose π_S is crystalline at all $v \mid p$ and let $T^*(1)R$ be as before. Choose $\widetilde{c} \in H^1(\pi_S, R)$ and let c be its in $H^1(\pi_S, R)$ under the split projection map $H^1(\pi_S, T^*(1)R) \to H^1(\pi_S, R)$. The tangent space $T_{\widetilde{c}}H^1(\pi_S, T^*(1)R)$ can be computed using the Lie algebra formalism, independently of the underlying geometric foundation, as

$$T_{\widetilde{c}}H^1(\pi_S, T^*(1)R) \cong H^1(\pi_S, L(c)^*(1) \times L(c)) \cong H^1(\pi_S, L(c)^*(1)) \times H^1(\pi_S, L(c))$$

The same formula above is true if we replaced $\widetilde{c}$ by a cocycle $\widetilde{c_v} \in H^1(\pi_v, T^*(1)R)$. Local Tate duality then implies

$$H^1(\pi_v, L(c_v)^*) \times H^1(\pi_v, L(c_v)) \cong H^1(\pi_v, L(c_v))^* \times H^1(\pi_v, L(c_v))$$

So $T_{c_v^*}H^1(\pi_v, T^*(1)R)$ has the structure of a symplectic vector space with symplectic form

$$\omega_v((f,c),(f',c')) = f \cdot c' - f' \cdot c$$

where $\cdot$ is the inner product structure on $T_{c_v^*}$. Summing over all $v \in S$ gives a symplectic form on $\prod_v T_{c_v^*}H^1(\pi_v, T^*(1)R)$. Abstractly, it follows that $\prod_v H^1(\pi_v, T^*(1)R) = \prod_v \mathcal{M}(\mathfrak{K}_v, T^*(1)R)$ is a $\mathbb{Q}_p$–analytic symplectic variety. Furthermore, Poitou–Tate duality implies $\mathrm{loc}_S(H^1(\pi_S, T^*(1)R))$ is a Lagrangian subvariety. Lastly, observe if $\widetilde{c_v}$ is crystalline or unramified, then $H^1_f(\pi_v, L(c_v)^*(1)) \times H^1_f(\pi_v, L(c_v))$ and $H^1(\pi_v/I_v, (L(c_v)^*(1))^{I_v}) \times H^1(\pi_v/I_v, (L(c_v))^{I_v})$ are Lagrangian inside $H^1_{ét}(\pi_v, L(c_v)^*(1)) \times H^1_{ét}(\pi_v, L(c_v))$. Hence

$$\prod_{v\in S} \mathcal{M}(\mathfrak{X}_v, T^*(1)R) \subset \prod_v H^1(\mathfrak{K}_v, T^*(1)R)$$

is a Lagrangian subvariety. Define

$$C(X,S;T^*(1)R) = \mathcal{M}(X^S, T^*(1)R) \times_{\prod_v \mathcal{M}(\mathfrak{K}_v, T^*(1)R)} \prod_v \mathcal{M}(\mathfrak{X}_v, T^*(1)R)$$

To conclude this discussion we note that the problem of realizing $C(X)$ as a Lagrangian intersection inside a symplectic manifold is solved whenever R is complex–semisimple. If R is also connected (e.g $SU(2)$[6] or $SL_2(\mathbb{C})$), then the absence of a Galois action implies that R is isomorphic to its Langlands dual. Kim points out that

> In the arithmetic setting, the self–dual nature of R is harder to arrange due to the existence of Tate twists.

To reconcile this issue, consider the case $R = GL_n(E)$. Then to achieve self–duality one could replace $\mathcal{M}(X^S, R)$ with $\mathcal{M}(X^S[\zeta_{p^\infty}], R) = H^1_{ét}(\pi_S^\infty, R)$, where

$$\pi_S^\infty = \pi_1^{ét}(\mathrm{Spec}(\mathcal{O}[S^{-1}, \zeta_{p^\infty}])$$

is the étale fundamental group of the affine scheme defined by the *cyclotomic extension* of $\mathcal{O}[S^{-1}]$. It would be interesting to study what role perfectoid geometry plays in these considerations.

[6] Recall that $SU(2)$ is isomorphic (in **Gp**) to the group of unit norm quaternions $\mathbb{H}^\times$, hence diffeomorphic (in **Top**) to S^3

Routes Towards Arithmetic Floer Homology

This section should not necessarily be taken as seriously as the last; however, it is psychologically pleasing that a reasonable notion of p–adic Floer homology likely exists. We first observe that the analogy between complex analytic spaces and adic spaces should actually be interpreted as an analogy between *almost complex-anaytic spaces* and adic spaces. Indeed,

Proposition 4. *An almost complex manifold X is globally complex if and only if the complex structures defined on TX glue together to a complex structure on X.*

If complex conjugation is "Frobenius as the prime ∞," then the above result stands in stark contrast to what is known in p–adic geometry; namely, that Frobenius at finite places never "jumps out" of its chart of definition. Hence Frobenius at finite places can only glue to a global Frobenius *map*, and not a *morphism*.

To define a p–adic Floer–type homology theory, we fist need to extend the notion of pseudoholomorphicity in definition 4 to situations in p–adic geometry. A–priori this is more delicate since the uniforization theory of p–adic curves is quite different than that of complex curves. The latter is well–known and summarized in Theorem 5; the former is the subject of [17].

Theorem 5. *Let D be a complex curve of (geometric) genus g. Then D admits a uniformization by a complex analytic manifold $\widetilde{D}$ given by*

$$\widetilde{D} := \begin{cases} \mathbb{P}^1_{\mathbb{C}} & g = 0 \\ \mathbb{A}^1_{\mathbb{C}} & g = 1 \\ \mathbb{H}^2 & g \geq 2 \end{cases}$$

The upshot is that there are several p–adic generalizations of the complex–analytic disk. The following list describes some potential routes of study towards p–adic analogue(s) of definition 4 using familiar machinery. Unless stated otherwise, X denotes some p–adic geometric object[7] defined over a mixed characteristic field K_v that is complete with respect to a v–adic nonarchimedian valuation. Briefly, the list below is an attempt to describe *the moduli problem of $\mathcal{T}$–Frobenius equivariant morphisms of p–adic curves*, where $\mathcal{T}$ denotes a sheaf on X.

1. Let $\mathcal{T}$ be a sheaf on X isomorphic to either the affinoid tangent sheaf $\mathcal{T}_{D_v}$ [2][§9] or some $\mathcal{D}$–module such as $\mathcal{D}^{\infty}_{D_v}$ from [26]. We define a smooth (resp. étale, syntomic) *pseudo-rigid-analytic* disk D_v^* to be a sm (resp. ét, syn) *$\mathcal{T}$-Frobenius equivariant* map $\phi^* : D_v \to X$, where $* \in \{\mathrm{sm}, \text{ét}, \mathrm{syn}\}$. In other words, a map from the rigid analytic 2–disk $\mathrm{Spm}(T^2(K_v))$ such that

   $$\mathrm{Fr}_X \circ \widetilde{\phi}_* = \widetilde{\phi}_* \circ \mathrm{Fr}_{D_v},$$

   Fr is the Frobenius endomorphism, and ϕ_* is the pushforward of $\mathcal{T}$. Given $x, y \in X$, $\widetilde{\pi}_{2,\mathrm{rig}}(x,y)^*$ denotes the collection of all pseudo-rigid-analytic curves D_v^* in X with marked points x and y. If D_v^* and C_v^* are pseudo-rigid-analytic curves that define the same object in $\mathcal{D}(\Delta^*(X))$ – the derived category of simplicial $*$–objects over X (i.e D_v^* and C_v^* are $*$–homotopic) – then we write $D_v^* \sim C_v^*$. Equivalence defined via homotopy categories are well–defined equivalence relations, so passing to the quotient space

   $$\widetilde{\pi}_{2,rig}(x,y)^* \to \pi_{2,rig}(x,y)^* =: \widetilde{\pi}_{2,rig}(x,y)^*/\sim$$

   yields the moduli space $\pi_2^{rig}(x,y)^*$, which classifies $*$–homotopy types of pseudo-rigid-analytic curves in X with marked points x and y. One can similarly construct the space of stable–homotopy classes of pseudo-rigid-analytic curves, denoted $\pi^{\mathrm{stab}}_{2,\mathrm{rig}}(x,y)^*$, using stable–homotopy equivalence.

2. A–posteriori there is no reason to restrict attention to rigid analytic spaces. In fact, the maximal spectrum of Tate algebras suppresses the arithmetic structure coming from nontrivial primes in the base field K_v. We can also define the notion of a *pseudo-Berkovich*, *pseudo-adic*, and *pseudo-perfectoid* closed

[7] e.g a formal scheme, rigid analytic space, Berkovich space, adic space, v–adic analytic manifold, perfectoid space, etc

disks by replacing the fundamental homological object $\mathrm{Spm}(K_v\langle T_1, T_2\rangle)$ by $\mathrm{SpB}(K_v\langle T\rangle)$, $\mathrm{Spa}(K_v\langle T\rangle, \mathcal{O}_v\langle T\rangle)$, and $\mathrm{Spa}(K_v\langle T^{\frac{1}{p^\infty}}\rangle, \mathcal{O}_v\langle T^{\frac{1}{p^\infty}}\rangle)$, respectively. We obtain families of moduli spaces

$$\pi_{2,\mathrm{rig}}(x,y)^*, \pi_{2,\mathrm{Berk}}(x,y)^*, \pi_{2,\mathrm{adic}}(x,y)^*, \pi_{2,\mathrm{Perf}}(x,y)^* \tag{9}$$

Where $*$ can in fact indicate any Grothendieck site $\mathcal{G}$. We anticipate the existence of comparison theorems between the above objects, much in the spirit of known comparison results between p–adic cohomology theories. Given $\pi_{2,\Xi}(x,y)^*$, where Ξ denotes some p–adic geometric model, we can try to imitate equation 5 to define a boundary map

$$\partial_\Xi(x) := \sum_y \sideset{}{'}\sum_{\phi^* \in \pi_{2,\Xi}(x,y)^*} |\mu(\phi^*)/K_v| \cdot y \tag{10}$$

where Σ' counts those pseudo-Ξ disks that are $\mathcal{T}$-Frobenius equivariant. In classical Floer homology theories, it is often exceptionally difficult to prove $\partial^2 = 0$. I am not certain that equation 10 gives the "right" definition, but this problem is considered to be of particular importance towards an arithmetic Floer *homology* theory.

2.2 Adelic Solenoids & Modular Braids

Let B_n and P_n denote the braid group on n strands and the pure braid group on n strands, respectively. Recall there exists a short exact sequence

$$1 \to P_n \to B_n \to S_n \to 1$$

Juyumaya and Lambropoulou define the (d,n)–modular framed braid group $\mathcal{F}_{d,n} := (\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n$ [12]. The p–adic framed braid group is the inverse limit

$$\mathcal{F}_{\mathbb{Z}_p} := \lim_{\leftarrow} \mathcal{F}_{p^r,n}$$

Geometrically, p–adic framed braids are p–braids whose framing is a p–adic integer. Similarly one may define the adelic framed braid group $\mathcal{F}_\mathbb{A} = \prod_p \mathcal{F}_{\mathbb{Z}_p}$.

Proposition 6. *Let Σ be a solenoid with a fixed embedding $\sigma : \Sigma \to S^3$. Then there exists an adelic braidword $\omega \in \mathcal{F}_\mathbb{A}$ such that Σ is the "closure" of ω.*

By the "closure" of an adelic braid, we mean that each knotted layer K_i in Σ is the closure of the braid at the i^{th} layer of ω.

TBC...

3 Elliptic curves, Alexander polynomials and zeta functions

In this section we recall some basic facts about the Dirichlet L–function attached to an elliptic curve defined over a $\mathbb{Q}$, and then compare the story to that of the Lefschetz zeta function associated to a knot (in S^3). Let $E/\mathbb{Q}$ be an elliptic curve of conductor N, p be a fixed prime, and $1_E(p) = \begin{cases} 1 & p \nmid N \\ 0 & p \mid N \end{cases}$ be the trivial character $\pmod N$. Define $t_1(E) = 2$ and $t_{p^n}(E) = p^n + 1 - |E(\mathbb{F}_{p^n})|$ for all $n > 1$.

Definition 7. The *counting zeta function* of E at p is

$$Z_p(E,z) = \exp\left(\sum_{n=1}^\infty \frac{t_{p^n}(E)}{n} z^n\right)$$

The well-known functional fact about $Z_p(E,z)$ is that

$$Z_p(E,z) = \left(1 - t_p(E)z + 1_E(p)\cdot pz^2\right)^{-1} \tag{11}$$

which is easily shown by taking the logarithmic derivative.

Definition 8. The L–function of E is defined as

$$L(E,s) \coloneqq \prod_p Z_p(E,p^{-s})$$

Remark 5. The significance of $L(E,s)$ is that it encodes the imformation $\{t_p(E)\}$ for *all* primes p. It can be shown that $L(E,s)$ converges absolutely on a right–half plane in $\mathbb{C}$. A holy–grail of modern number theory is to understand what happens at $s=1$.

Proposition 7. *$L(E,s)$ admits a summation formula*

$$L(E,s) = \sum_{n\geq 1} a_n n^{-s}$$

such that $a_1 = 1$, $a_p = t_p(E)$, $a_{p^i} = a_p a_{p^{i-1}} - 1_E(p)pa_{p^{i-2}}$ for $i \geq 2$, and $a_{mn} = a_m a_n$ for relatively prime m and n. Conversely, given any integer sequence $\{a_i\}_{i\geq 1}$ satisfying the aforementioned properties, the Dirichlet series $\sum_{n\geq 1} a_n n^{-s}$ admits an Euler product formula

$$\sum_{n\geq 1} a_n n^{-s} = \prod_p \left(1 - a_p p^{-s} + \chi(p)\cdot p^{1-2s}\right)^{-1}$$

Proof. This result is an application of the unique factorization of $\mathbb{Z}$ and a basic exercise in algebraic manipulation, so we omit the details. ☐

Definition 9. Let X be a manifold such that $H_*(X,\mathbb{Q})$ is a finite dimensional $\mathbb{Q}$–vector space. For any continuous map $t : X \to X$ one can define a *Lefschetz zeta function* for t by the formula

$$\zeta_t(z) \coloneqq \exp\left(\sum_{n=1}^{\infty} \frac{\Lambda(t^n)}{n} z^n\right)$$

where $\Lambda(t) = \sum_{i=0}^{\dim X} (-1)^i \mathrm{Tr}(t_{*i} : H_i(X,\mathbb{Q}) \to H_i(X,\mathbb{Q}))$ is the Lefschetz number of t.

One can also define a *relative* and *restricted* versions the Lefschetz zeta function as follows.

Definition 10. Let X and P be a manifolds such that $H_*(P,\mathbb{Q})$ and the relative homology $H_*(M,P;\mathbb{Q})$ are finite dimensional $\mathbb{Q}$–vector spaces. For any continuous map $t : (M,P) \to (M,P)$ we can define a *relative* zeta function by

$$\zeta_{t;P}(z) \coloneqq \exp\left(\sum_{n=1}^{\infty} \frac{\Lambda_{M;P}(t^n)}{n} z^n\right)$$

where $\Lambda_{M;P}(t) = \sum_{i=0}^{\dim M} (-1)^i \mathrm{Tr}(t_{*i} : H_i(M,P;\mathbb{Q}) \to H_i(M,P;\mathbb{Q}))$ is the *relative* Lefschetz number of t.

The *restricted* Lefschetz zeta function $\zeta_{t|P}$ is given by

$$\zeta_{t|P}(z) \coloneqq \exp\left(\sum_{n=1}^{\infty} \frac{\Lambda((t \mid P)^n)}{n} z^n\right)$$

where $\Lambda(t \mid P) = \sum_{i=0}^{\dim P} (-1)^i \mathrm{Tr}((t \mid P)_{*i} : H_i(P,\mathbb{Q}) \to H_i(P,\mathbb{Q}))$ is the *restricted* Lefschetz number.

The utility of these definitions is illustrated by the following Proposition, which is Lemma 10 of [21].

Proposition 8. *If at least two of the groups $H_*(X,\mathbb{Q})$, $H_*(P,\mathbb{Q})$, and $H_*(X,P;\mathbb{Q})$ are finite dimensional, then ζ_t, $\zeta_{t;P}$, and $\zeta_{t|P}$ are defined and*

$$\zeta_t(z) = \zeta_{t;P}(z) \times \zeta_{t|P}(z)$$

Proof. The long exact sequence on relative homology

$$\cdots \to H_i(P) \to H_i(X) \to H_i(X,P) \to \cdots$$

gives

$$\Lambda(t) = \Lambda_{M;P}(t) + \Lambda(t \mid P)$$

The formula in the proposition immediately follows. □

Noguchi uses the product formula in Propoition 8 to prove:

Theorem 9 (Noguchi)**.** *Let X be a compact connected n–manifold, X_∞ an orientable infinite–cyclic covering of X with $\dim_{\mathbb{Q}} H_*(X_\infty,\mathbb{Q}) < \infty$, and $t : (X_\infty, \partial X_\infty) \to (X_\infty, \partial X_\infty)$ a proper continuous map of degree $\lambda \neq 0$ with respect to $H^*_{cpt}(X_\infty; \partial X_\infty, \mathbb{Q})$. If n is odd, then the Lefschetz zeta functions ζ_t and $\zeta_{t|\partial X_\infty}$ satisfy the functional equation*

$$\frac{\zeta_t(\frac{1}{\lambda z})^2}{\zeta_{t|\partial X_\infty}(\frac{1}{\lambda z})} = \lambda^\chi z^{2\chi} \frac{\zeta_t(z)^2}{\zeta_{t|\partial X_\infty}(z)}$$

where χ is the Euler characteristic of X_∞. In particular, when $\partial X_\infty = \emptyset$ one has

$$\zeta_t(\frac{1}{\lambda z}) = \pm \lambda^{\frac{\chi}{2}} z^\chi \zeta_t(z)$$

Corollary 10. *When X_∞ is the infinite cyclic covering of a (tame) knot complement $X = S^3 \backslash K$ and t is the monodromy map induced by a generator τ for $\mathrm{Gal}(X_\infty/X)$, then*

$$\zeta_t(z) = \frac{1}{\Delta_K(0)} \frac{\Delta_K(z)}{1-z}$$

where $\Delta_K(t)$ is the Alexander polynomial of K, normalized so that all powers of z are positive and Δ_K is of minimal degree.

TBC...

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