Recall that if N is any positive integer, we can write it as a product of its prime factors:

$$N = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_n^{a_n}$$

Suppose that we write this product in such a way that the prime number $p_1, p_2, \ldots, p_n$ are in *increasing* order according to size; in other words, we assume $p_1 < p_2 < \cdots < p_n$. Then we can associate two *instrinsic quantites* to N, which we call $\alpha(N)$ and $\beta_+(N)$:

$$\alpha(N) = \frac{(a_1 \times p_1) + (a_2 \times p_2) + \cdots (a_n \times p_n)}{a_1 + a_2 + \cdots a_n}$$

$$\beta_+(N) = p_n - \cfrac{1}{p_n - \cfrac{1}{\ddots - \cfrac{1}{p_{n-1} - \cfrac{1}{p_{n-1} - \cfrac{1}{\ddots \cfrac{1}{p_1 - \cfrac{1}{p_1}}}}}}}$$

where there are a_n copies of p_n, a_{n-1} copies of p_{n-1}, …, and a_1 copies of p_1 appearing in the *continued fraction* expression for $\beta_+(N)$ above.

Definition 1. We say that a positive integer N is *flat* if $\alpha(N) = \beta_+(N)$, and we call N *sharp* if $\beta_+(N) > \alpha(N)$.

Example 1. Consider $N = 12$. Then $N = 2^2 \times 3$ so

$$\alpha(12) = \frac{(2 \times 2) + 3}{3} = \frac{7}{3} = 2\frac{1}{3}$$

and

$$\beta_+(12) = 3 - \frac{1}{2 - \frac{1}{2}} = 3 - \frac{2}{3} = \frac{7}{3} = 2\frac{1}{3}$$

So $\beta_+(12) = \alpha(12)$, which means that 12 is a flat number.

Example 2. Consider $N = 8$. Then $N = 2^3$ so

$$\alpha(8) = \frac{3 \times 2}{3} = 2$$

and

$$\beta_+(8) = 2 - \frac{1}{2 - \frac{1}{2}} = 2 - \frac{2}{3} = \frac{4}{3} = 1\frac{1}{3}$$

So we see that $\beta_+(8) < \alpha(8)$, which means that 8 is neither sharp nor flat.

1: Decide if the following numbers are *flat*, *sharp*, or neither.

(a) 4
(b) 6
(c) 9
(d) 10
(e) 15
(f) 20
(g) 24
(h) 30
(i) 36
(j) 45
(k) 48

2: Let p be any prime number. Prove that p is flat.

3: Now let N be any number that has a prime factorization of the form $N = p^2$ where p is one prime number (E.g. $4 = 2^2$, $9 = 3^2$, $25 = 5^2$). Prove that n is neither flat nor sharp.

4: Make an educated guess about which numbers are flat (an educated guess means that you can give an argument for why the guess seems correct)

At first there is no reason why we should define $\beta_+(N)$ so that the first prime in the continued fraction expression (p_n) is the largest. We can also define

$$\beta_-(N) = p_1 - \cfrac{1}{p_1 - \cfrac{1}{\ddots - \cfrac{1}{p_2 - \cfrac{1}{p_2 - \cfrac{1}{\ddots \cfrac{1}{n - \cfrac{1}{p_n}}}}}}}$$

where $p_1 < p_2 < \cdots p_n$ as before. We could then say that N is *anti–flat* if $\alpha(N) = \beta_-(N)$, and N is *anti–sharp* if $\alpha(N) < \beta_-(N)$.

5: Make an argument for why the notions of anti–flatness and anti–sharpness are not very interesting. (Hint: are there any anti–sharp numbers?)