ARITHMETIC, GEOMETRY, AND MATHEMATICAL REALITY

THE NON–UNIQUENESS OF SPATIOTEMPORAL PERCEPTION

Contents

Abstract. We show how p–adic arithmetic and geometry pose issues for Kant's epistemology and Mill's empricism as they relate to mathematical knowledge. Recent trends in phyiscs, biology, and economics include using p–adic analysis to develop accurate theories of complex systems that have been difficult to study using traditional techniques in real analysis. The apparent success of these theories suggests that our spatiotemporal intuitions (in the sense of Kant) are significantly limited when it comes to acquiring mathematical knowledge. Furthermore, it may be necessary to revise our understanding of what it means to empirically observe mathematical phenomena (in the case of Mill).

1. Introduction

Developments in mathematics, such as non–Euclidean geometry, and the subsequent application of these ideas to physics pose problems for Kant's spatiotemporal–dependent theory

Date: 18 May, 2018

Word count: ≈ 3200.

of mathematical knowledge and Mill's emprical outlook on mathematics. To Kant, there are four characterizations of all knowledge. *A priori* (necessary) knowledge is that which exists independent of observation. *A posteriori* (empirical) knowledge is obtained when sensory apparatus, e.g. sight and touch, are presupposed to make judgments of reality. *Analytic* knowledge includes all statements which can be deduced by understanding the statement itself. For instance, 'all mice are rodents' is an analytic statement since the concept 'rodent' is "contained in the concept" of 'mice' [Kan16, pp. 225]. In contrast, *synthetic* knowledge requires the conscious observer to go beyond simply understanding language. Table 1 is a summary of Kant's epistemology. Kant believed that mental constructions of space and time are a priori facets of both mathematical knowledge and emprical experience. In particular, the way in which we perceive space and time, which is independent of sensory observation, gives rise to our understanding of geometry and arithmetic, respectively.

Modern physics, however, tells us that our perception of spacetime is dependent on mass, acceleration[1], and the velocity of objects relative to a stationary observer. For instance, if we were standing in the center of a disk rotating so that the perimeter is near the speed of light, then the disk would appear non–Euclidean [2]. Somewhat similarly, Mill believed

Table 1. Kant's Epistemology

	a priori	a posteriori
analytic	logic	–
synthetic	mathematics	"judgments of experience"

that mathematics makes claim about the physical world. Axioms and theorems in geometry and arithmetic are true insofar as we can observe their truth value, either approximately (in the case of geometry) or exactly (for arithmetic), in what Kant calls the "world of experience" [Hit18]. The parallel postulate in Euclidean geometry, for instance, is validated through observing many instances of two parallel lines never meeting, and then infering the result is always true by heuristic induction.

[1] $F_{grav} = ma$

[2] Length contractions at the perimeter imply that the ratio of circumference to diameter would be apparently not equal to π

In the present paper we argue for the claim that Kant and Mill's ignorance of more advanced areas of mathematics contributed to incompleteness in Kant's theory of knowledge and Mill's empiricist viewpoint. We will highlight their unawareness of a different system of arithmetic – the p–adic numbers. For Kant's theory, p–adic arithmetic introduces a hole in the viewpoint that any being with the capacity to obtain knowledge "must do so in the same way" [Hit18]. For Mill, p–adic numbers calls into question to what extent can certain aspects of mathematics be empirically observed; and, when p–adic numbers have representations in the observed world, to what degree do such representations constitute empirical evidence.

The example of p–adic numbers are used since they are the prototype of *non–Archimedean* numbers, and their relation to the real numbers is an arithmetic analogue of the duality between Euclidean and non–Euclidean geometry. The use of p–adic arithmetic in physics was inspired by the following quote of Yuri Manin.

> *On the fundamental level our world is neither real, nor p-adic, it is adelic. For some reasons reflecting the physical nature of living matter (e.g. the fact that we are built of massive particles), we tend to project the adelic picture onto its real side. We can equally well spiritually project it upon its non-Archimedean side and calculate most important things arithmetically.* [Man87]

In section 2 we will review Kant and Mill's epistemology. Section 3 includes an elementary introduction to p–adic and adelic numbers, and we discus the philosophy of these arithmetic systems in relation to Kant and Mill's philosophy of mathematics. We do not presuppose any familiarity with these number systems, and only cover the minimum necessary for the subsequent exposition.[3] In section 4 we outline several advances in the sciences that utilize p–adic numbers. This leads us to either reject Kant and Mill's account of mathematics or we must revise Kant's theory to include the possibility of knowledge–acquiring beings who experience space and time p-adically, and revise Mill's theory to include the possibility of

[3]An understanding of the real number system is sufficient to understand the definition of p–adic numbers

empirically observing what we call *non–physical observable phenomena* and *physical non–observable phenomena*.

Warning. Our goal is not to argue for the existence of an adelic universe; but rather, we present a specific scenario that suggests such a framework *could* exist outside the scope of human cognition, and then discuss the philosophical ramifications on Kantian and Millian epistemology *if* the universe is adelic.

$\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ will denote the sets of whole numbers, rational numbers, and real numbers, respectively.

2. Kant and Mill's philosophy of mathematics

Kant's theory of knowledge purports that mathematics is a priori synthetic knowledge [Kan16, pp 226], meaning that mathematics consists of necessary truths that can be deduced via reasoning independent of sensory experience. In general, all knowledge is related to possible mental states, which Kant calls "intuitions" [Kan16, pp 239]. Knowledge cannot arise unless the mind unifies the image of the world it takes in, i.e. the "intuition of form," with its ability to process said input into meaningful sensibilities, i.e. "intuitions of phenomena" [Kan16, pp. 228]. When one strips away mental impressions of phenomena, as well as all sensory data that our mind associates to these impressions, one is left with "pure intuitions." Kant argues that the only pure intuitions of form are space and time, and these intuitions admit all forms of empirical knowledge. Moreover, intuitions of space and time give rise to the synthetic a priori intuitions of geometry and arithmetic, respectively. In particular, the forms of space and time are independent of the knowledge–acquiring observer.

In retrospect, the requirement that all knowledge–acquiring beings interpret space and time in the same fashion seems naïve and anthropocentric. Especially considering that Kant believed space was Euclidean [MM16, pp. 220], while a consequence of Einstein's theories is a non–Euclidean spacetime.

Mill believed that mathematics is inherently empirical, rejecting Kant's claim that mathematics is a priori knowledge in the sense that it can be obtained outside of observational phenomena. Mathematical axioms are formulated and taken for granted since "they rest on superabundant and obvious evidence" [Mil16, pp. 267]. The substantive propositions of arithmetic are known through "early and constant experience" [Mil16, pp. 271]. Mill gives the example that we know $2+1=3$ by studying objects in the physical world, but without empirical data to confirm this equality, we would have no basis to believe it true. Furthermore, only by recognizing objects in the physical world does Mill believe a definition of natural number can be made. Each $n \in \mathbb{N}$ only has meaning if it is interpreted as the aggregate of multiple objects of a given kind, e.g. n pebbles, n fingers, etc. While not implicitly stated, a similar definition for real numbers can be found in Mill's work when he discusses interpreting geometric magnitudes as weights, lengths, or areas of physical objects. In general, Mill states that all geometric knowledge is an approximate description of physical truths. In his words, "every theorem in geometry is a law of external nature" [Mil16]. Ideal objects such as straight one–dimensional lines and perfect circles do not exist in reality; however, all objects in geometry are simplifications of real objects, and since geometric objects to not imply any false properties about real objects, we are able to obtain true statements about the physical world by studying geometric truth. Up until 1897, all arithmetic and geometry occurred within the confines of the *real* number system $\mathbb{R}$. The development of the p–adic number system calls into question certain aspects of both Kant and Mill's theories. In what follows we define the p–adic numbers, and discuss what effects their existence has on Kantian and Millian epistemology.

3. Spatiotemporal philosophy of the p–adic and adelic numbers

p always denotes some prime number. In calculus[4], one learns the following definition.

Definition 1. A sequence of rational numbers $x_1, x_2, \ldots$ is Cauchy if for all $\epsilon > 0$ there exists $N > 0$ such that $|x_m - x_n| < \epsilon$ for all $m, n \geq N$

[4]E.g. Caltech's Math 1a

Moreover, $\mathbb{R}$ is *a Cauchy completion* of $\mathbb{Q}$. This means the set $\mathbb{R}$ is obtained by demanding that all Cauchy sequences in $\mathbb{Q}$ converge to an actual number. There is a subtlety in this construction; namely, the definition of a Cauchy sequence depends on our notion of distance. Two rational numbers x and y are "close" if $|x - y|$ is "small." Kant would argue that this statement is a priori synthetic knowledge derived from pure intuitions about space. Mill would say that the statement is true by virtue of experience. Just as Kantian spatial intuitions or Millian existential empiricism give rise to the Euclidean picture of the universe, they also imply our usual *Archimedean* system of arithmetic. However, the scale and reference–frame of our existence affects our ability to gauge sizes and shapes of tangible objects. If humanity existed at the quantum level, or if were born in a world with drastically non–uniform gravity[5], then our notions of relative size and distance would surely be different. Only after the notion of distance is formally defined does one realize our familiar *metric* (see definition 2 in the Appendix) is not unique. This observation supports Kant's claim that mathematics is synthetic knowledge, but, as we illustrate in what follows, the non–uniqueness of metrics poses problems for the Kantian necessity of spatiotemporal perception and the Millian reliance on empirical observation.

Every rational number q can be written *uniquely* in the form $q = p^v \cdot \frac{a}{b}$ for some $a, b, v \in \mathbb{Z}$ such that p does not divide a nor b. Define the map $|-|_p : \mathbb{Q} \to \mathbb{R}$ via $|q|_p = p^{-v}$. $|-|_p$ satisfies properties (1)–(4) and (4^+) in definition 2, meaning it defines a *non–Archimedean* metric[6]. So we can take the Cauchy completion of $\mathbb{Q}$ using $|-|_p$ instead of $|-|$ as our notion of distance. The resulting set is denoted $\mathbb{Q}_p$ and called the *p–adic rational numbers*. Since $|-|_p$ is non–Archimedean, $\mathbb{Q}_p$ is a non–Archimedean space.

One can regard $\mathbb{Q}_p$ as the set $\mathbb{R}$ equipped with an exotic metric. Two numbers $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathbb{Q}_p$ are "close" if $\mathfrak{a} - \mathfrak{b}$ can be evenly divided by p^v for some large $v \in \mathbb{Z}$. To illustrate, suppose $p = 3$ and consider the 3–adic numbers $\mathfrak{a} = 1$ and $\mathfrak{b} = 3^{1000000000} + 1$. Then $|\mathfrak{a} - \mathfrak{b}|_3 = \frac{1}{3^{1000000000}}$, so 1 and $3^{1000000000} + 1$ are very close 3–adically despite being far apart in the usual sense. Albeit confusing, on a mathematical level there is nothing special about

[5]So that light rays from nearby sources appear curved instead of straight

[6]Ostrowski's Theorem says that the only metrics that can be put on $\mathbb{Q}$ are $|-|$ and the p–adic metrics $|-|_p$

$\mathbb{R}$. There are p–adic theories of arithmetic, geometry, calculus, and differential equations: see [Ser79, Moc91, Sch85, Ked10], respectively. So what is the use of $\mathbb{Q}_p$ outside of pure mathematics? If knowledge of $\mathbb{Q}_p$ is a priori synthetic, then $\mathbb{Q}_p$ is necessary to the structure of Kant's "world of experience" [Hit18]. And if mathematics is empirical, qua Mill, then it should be possible to observe p–adic arithmetic and geometry in the physical world.

One of the main criticisms of Kant and Mill's theories is that they do not admit the possibility of non–Euclidean geometries. Without knowledge of $\mathbb{Q}_p$, this tacitly means non–Euclidean geometries where the points live in Archimedean space ($\mathbb{R}$ or $\mathbb{C}$). Hence Kant and Mill's account of mathematics does not admit the possibility that the base space is non–Archimedean, let alone that the universe might have non–Euclidean geometry *and* non–Archimedean arithmetic. Based on our spatiotemporal experience of reality, non–Archimedean fields seem too strange to even posit the possibility that our world has such structure. After–all, if the world were, say 3–adic, and I take 3^2 one-foot steps in any direction, I would expect to move a distance $\frac{1}{3^2}ft$ from the starting position. Spatiotemporal intuitions indicate that this is clearly not the case. But, in principle, anything that can be said about $\mathbb{R}$ has an analogue in $\mathbb{Q}_p$, so we arrive at a strange paradox. On one hand, we believe that mathematics over $\mathbb{R}$ models the physical world to incredible accuracy, and our intuitions of phenomena/empirical data tell us that the world is not p–adic. On the other–hand, from pure mathematical point of view $\mathbb{R}$ is not distinguished over $\mathbb{Q}_p$. This dichotomy motivates the following questions.

> Do the p–adic numbers describe a world external to our intuitions and senses?
> To what extent can we *know* that the universe should be well–modeled *only* by the real numbers?

Our discussion in the next section attempts to answer each of these questions. If there is a p–adic world external to our sensory apparatus, then which prime p is it defined by? Yuri Manin suggests "it is more reasonable to believe in a democracy of all available choices in metric" [Man87]. So we collect all of these possible worlds into a unified model of arithmetic by defining an infinite Cartesian product:

$$\widetilde{\mathbb{A}} = \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{Q}_5 \times \cdots \times \mathbb{Q}_p \times \cdots$$

$\widetilde{\mathbb{A}}$ is called the "big set of adeles," or just the *adeles*[7]. Manin's philosophy is that Kant's "world in of itself" is Adelic, but our "world of experience" is the projection of $\widehat{\mathbb{A}}$ onto its $\mathbb{R}$ coordinate. In this framework there is no reason to reject the existence of worlds that are projections onto some $\mathbb{Q}_p$ coordinate. The beautiful formula motivating Manin's viewpoint is

$$|q| \times \prod_{p\text{: prime}} |q|_p = 1 \text{ for all rational } q \neq 0$$

Hence whenever a big empirical measurement is made (i.e. $|q| > 1$), then at least one of the corresponding p–adic lengths should be small (i.e. $|q|_p < 1$). If the unit 1 is taken to be on the atomic scale, then the $\mathbb{Q}_p$ worlds could be those that exist at the quantum level. If this were the case, then the existence of p–adic arithmetic at quantum scales could explain why we have results such as the Heisenberg Uncertainty Principles: we fail to have complete information (over $\mathbb{R}$) about quantum dynamics since the phenomena at those scales actually exists over $\mathbb{Q}_p$.

Mill's empiricism cannot directly account for this possibility since the nature of our observed reality prohibits exact observation of quantum phenomena. In general, there is no obvious physical object in the world that verifies, in a Millian sense, mathematical truths about $\mathbb{Q}_p$. Note how the existence of non–Archimedean arithmetic provides a more serious error for Mill's empiricism than non–Euclidean geometry. If Mill were alive today, he could easily point to large black holes that curve spacetime to empirically verify non–Euclidean geometry. However, as is illustrated in the next section, there are no known tangible objects that behave according to the rules of p–adic arithmetic. Kant's spatiotemporal necessity also breaks down under these considerations. For Kant, our understanding of geometry and arithmetic, qua $\mathbb{R}$, is derived from a priori intuitions of space and time. But where does our understanding of mathematics

[7]A special subset of $\widehat{\mathbb{A}}$ is what mathematicians usually call the Adeles, but for simplicity we work with all of $\widehat{\mathbb{A}}$

qua $\mathbb{Q}_p$ come from? While there is no problem declaring that knowledge about $\mathbb{Q}_p$ is a priori synthetic, such knowledge cannot be the result of spatiotemporal intuitions. If it were, then we would necessarily construct spatiotemporal experience within a p–adic framework, or at least within a framework combining $\mathbb{R}$ and some $\mathbb{Q}_p$.

4. p–ADIC MATHEMATICS IN THE UNIVERSE

Despite the fact that there are no known physical objects that are well–modeled by arithmetic and geometry over $\mathbb{Q}_p$, there are several examples of non–physical ideas and systems across science that behave according to p–adic principles. To begin positing examples, we must ask: can one visualize p–adic vector spaces? The answer is akin to asking if we can visualize non–Euclidean geometries. Figures 1 and 2 show pictorial representations of $\mathbb{Q}_2$ and $\mathbb{Z}_3$[8], respectively. Each node in 2 represents a 3–adic whole number. A posteriori, these figures help identify non–physical phenomena in our "mode of experience" [Kan16, pp 235] that have p–adic properties. Consider figure 3. The evolutionary tree has structure similar to a p–adic tree. Recently, biologists have found it useful to model DNA and how closely two species are related using approximations with $|-|_p$ [Dra12]. In the same spirit, Zharkov developed an adelic theory of the stock market in which he uses certain 3–adic fractal curves to obtain strikingly accurate models of real markets. Figure 4 shows one such curve. p–adic methods have also been useful in modern physics. Within the past 30 years, p–adic models of quantum mechanics, string theory, and dark matter/energy have been developed [Var11, Dra06]. In what sense do these three examples constitute physical evidence for the existence of $\mathbb{Q}_p$? The example of the evolutionary tree and the stock market are best characterized as *non–physical observable phenomena*. They exist in reality, not as tangible objects, but as well–defined processes that can be studied via other senses. The subject matter of quantum mechanics, string theory, and dark matter are all thought to be true physical objects. Quantum mechanics studies the behavior collections of subatomic particles, string theory's fundamental objects

[8]See the appendix for the definition of $\mathbb{Z}_3$ – the 3–adic whole numbers

are tiny oscillating strings that are thought to be the most basic building–blocks of all matter, and dark matter is theorized to be composed of an exotic physical particle that does not interact with light. In each case the physics studies *physical non–observable phenomena*.

In traditional Millian terms, the evolutionary tree and the stock market are not physical objects, nor are they attributes of physical objects. In particular, the fact that these systems can be well–modeled by $\mathbb{Q}_p$ does not necessarily empirically confirm the existence of $\mathbb{Q}_p$. To justify arithmetic in $\mathbb{N}$ (resp. $\mathbb{R}$), Mill points to collections (resp. lengths and weights) of physical objects and claims that the numerical size of a collection (resp. lenght or weight) is an attribute of the objects. An analogous treatment of arithmetic in $\mathbb{Q}_p$ is not feasible. There are no physical $\mathbb{Z}_p$–aggregates of pebbles, nor is there bullion whose weight is recorded as a number in $\mathbb{Q}_p$. Therefore, to justify arithmetic in $\mathbb{Q}_p$, one must either reject Mill's empiricism entirely or revise the notion of empiricial observation to account for *non–physical observable phenomena*. Similarly, the nature of subatomic particles, strings, and dark matter does not admit the possibility of exact observation. The former and the latter are detected by indirect methods, yet in all three cases the objects of study can never be observed in the same way that the fingers on our hands or the shape of the earth can be observed. It follows that Mill's empiricism must be rejected or our notion of empirical observation must be revised to also account for *physical non–observable phenomena*. Revising the notion of what constitutes empirical observation is related to the issue of humans having a limited capacity for spatiotemporal perception.

Kant's contention is that any being with the capacity for knowledge must possess the same intuitions of space and time. These pure intuitions are the basis for all forms of knowledge, yet arithmetic and geometry over $\mathbb{Q}_p$ exists beyond our spatiotemporal perceptions. Even when we draw representations of $\mathbb{Z}_p$ or $\mathbb{Q}_p$ like in figures 1 and 2, we cannot capture a completely accurate "intuition" of what these numbers look like. Indeed such drawings are always made on a region of $\mathbb{R}^2$ or $\mathbb{R}^3$, so we inevitably artificially intuit $\mathbb{Q}_p$ in the framework of *real* spatiotemporal intuitions. Since $\mathbb{R}$ is not mathematically more special than $\mathbb{Q}_p$, it is not unreasonable to hypothesize a p–adic world, where, for instance, pictorial representations of the real line $\mathbb{R}$ have to be superimposed onto $\mathbb{Q}_p^2$. Or where the configuration of our hands and fingers look like a $\mathbb{Z}_2$ tree,[9] and instead of counting like $1, 2, 3, 4$, etc. we count like

$$(1,0,0,0,\dots), \quad (0,1,0,0,\dots), \quad (0,0,1,0,\dots), \quad (0,0,0,1,0,\dots), \quad \cdots$$

i.e. how one counts in $\mathbb{Z}_2$. It is difficult to imagine p–adic intuitions of space and time, but only by virtue of how our minds processes, what Kant calls, "intuitions of phenomena". In other words, while Kantian philosophy places our notion of space and time within the category of pure intuitions of form, the fact that knowledge about $\mathbb{Q}_p$ is not accurately reflected in our *real* spatiotemporal experiences indicates that we must either reject Kant's claim that all beings with the capacity to acquire knowledge intuit space and time the same way, or else we must revise Kantian epistemology to include the possibility of beings who process space and time according to the laws of $\mathbb{Q}_p$. One way to achieve the latter is to declare that spatiotemporal perceptions, i.e. judgments about space and time, are in fact another form of knowledge; namely, a priori synthetic knowledge. In Millian terms, our knowledge about the nature of space and time is acquired by continuous exposure since the moment we are born. This revision is illustrated in table 3.

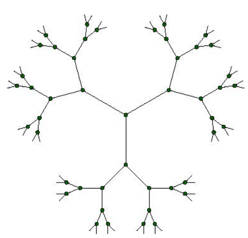

FIGURE 1. Bruhat–Tits tree for the 2–adic "sphere": if we could continue the Fractal–pattern indefinitely, the points in the limit of the tree are bijective with $\mathbb{Q}_2$.

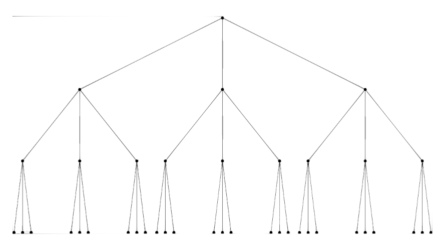

FIGURE 2. Standard planar tree of $\mathbb{Z}_3$: if we could continue the tree indefinitely, the nodes are bijective with elements of $\mathbb{Z}_3$.

[9]Figure 2 with two prongs extending downwards at each node instead of three

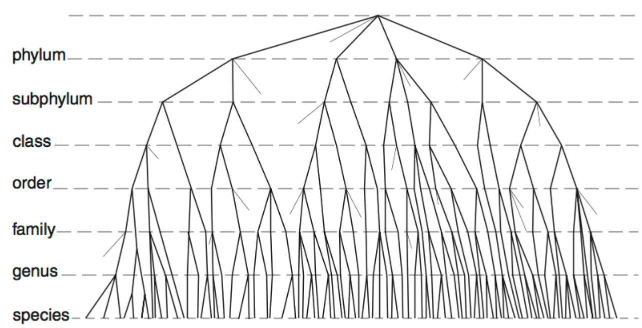

FIGURE 3. A simplification of a well–known tree diagram from evolutionary biology showing the evolutionary history of life on Earth. Image sourced from [EH01]

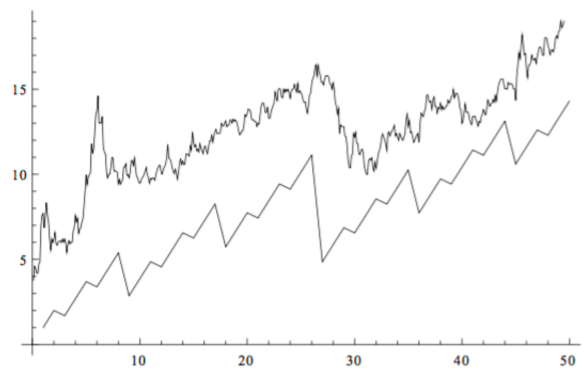

FIGURE 4. One of Zharkov's Fractal 3–adic curves (top) compared against the real data from the Russian stock Index (bottom). Image sourced from [Zha11]

5. Conclusion

Kant's theory purports that pure intuitions of space and time are a priori in the sense that they admit our intuitions of geometry and arithmetic, as well as all empirical forms of knowledge. We argued that knowledge of p–adic geometry and arithmetic that exists outside our *real* spatiotemporal framework offers evidence against Kant's assumption of the spatiotemporal–dependence of mathematical knowledge. If table 2 accurately reflects the nature of the "world of experience," then Kant's theory of epistemology only accounts for the top–left box. Mill's account of mathematics insists that geometry and arithmetic are verified by observations in the physical world. Our arguments show that if this viewpoint is to be taken seriously, then a revision of our notion of empirical knowledge is necessary. While we may never be able to literally see p–adic geometries, or exactly probe the universe at sub–quantum scales, the fact that several phenomena across the sciences can be accurately modeled by p–adic theories hints that Yuri Manin's assertion about an adelic universe may have more validity than traditional modern science and philosophy professes.

Table 2. Spatiotemporal Dependendene of Modes of Cognition

	Non–relativistic speeds	Relativistic speeds
Human scale	Euclidean geometry and Archimedian arithmetic	non–Euclidean geometry and Archimedian arithmetic
Quantum scale	Euclidean geometry and non–Archimedian arithmetic	non–Euclidean geometry and non–Archimedian arithmetic

Table 3. Modern Kantian Epistemology

	a priori	a posteriori
analytic	logic	judgments of space and time
synthetic	mathematics	"judgments of experience"

Appendix A. Mathematical Refresher

Definition 2. A *metric* (on $\mathbb{Q}$) is determined by a function $|-| : \mathbb{Q} \to \mathbb{R}$ with the properties

(1) $|x| \geq 0$ for all $x \in \mathbb{Q}$
(2) $|x| = 0$ if and only if $x = 0$
(3) $|xy| = |x||y|$ for all $x, y \in \mathbb{Q}$
(4) $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{Q}$ (triangle inequality)

We can also replace (4) with a stronger condition:

4^+: $|\mathfrak{a} + \mathfrak{b}|_p \leq \max\{|\mathfrak{a}|_p, |\mathfrak{b}|_p\}$ for all $\mathfrak{a}, \mathfrak{b} \in \mathbb{Q}_p$ (non–Archimedean ultrametric inequality)

If $|-|$ satisfies (1)–(4) but not (4^+), it is called *Archimedian.* If it moreover satisfies (4^+), $|-|$ is known as *non–Archimedian.*

In our familiar decimal system of arithmetic in $\mathbb{R}$, every real number r can be written in the form

$$r = \sum_{n=-\infty}^{N} r_n \cdot 10^n = r_N \cdot 10^N + r_{N-1} \cdot 10^{N-1} + \cdots + r_1 \cdot 10^1 + r_0 + r_{-1} \cdot 10^{-1} + \cdots$$

such that each r_i is an integer, $0 \leq r_i < 10$, and 10^N is the largest power of 10 smaller than r. For example, if $r = 33.3$, then $r = 3 \cdot 10^1 + 3 + 3 \cdot 10^{-1}$ and $N = 1$. Observe that $\mathbb{Z}$ is recovered as the set of r where $r_i = 0$ whenever $i < 0$ in the above expansion. Let $q_i = r_i \cdot 10^i$, and note the set of truncated sums

$$\{q_N, q_N + q_{N-1}, q_N + q_{N-1} + q_{N-2}, \dots\}$$

defines a Cauchy–sequence of rational numbers. For example, the sequence $\Pi = \{4, 4 - \frac{4}{3}, 4 - \frac{4}{3} + \frac{4}{5}, 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7}, \dots, \sum_{n=0}^{N} (-1)^n \frac{4}{2n+1}, \dots\}$ consists entirely of rational numbers and defines a Cauchy sequence. However, the limit of Π is not a rational number, suggesting there is some "gap" in $\mathbb{Q}$. To fill in this gap, mathematicians abstractly declare that Π converge to a number. Π turns out to converge to π. So the abstract procedure of Cauchy completion returns fundamental numbers associated to (approximations of) our intuitions of the real world, such as $\sqrt{2}$, π, e, φ, and all other irrational quantities. In parallel with real numbers, every p–adic rational number $\mathfrak{a}$ can be written in the form

$$\mathfrak{a} = \sum_{n=N}^{\infty} \mathfrak{a}_n \cdot p^n$$

where $N \in \mathbb{Z}$ and the $\mathfrak{a}_n$ are integers not divisible by p.

Definition 3. those $\mathfrak{a}$ so that $\mathfrak{a}_i = 0$ whenever $i < 0$ form a set $\mathbb{Z}_p$ known as the *p–adic integers.*

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